97. *Note on the Lexicographic Product of Ordered Semigroups*

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A semigroup S with a simple order $\leqq$ is called a *left* [*right*] *ordered semigroup* if it satisfies the condition that

for every $x, y, z \in S$, $x \leqq y$ *implies* $zx \leqq zy$ [$xz \leqq yz$].

S is called an ordered semigroup if it is a left and right ordered semigroup. Let $\{S_\alpha ; \alpha \in A\}$ be a collection of semigroups, each of which has a simple order and let the index set A be a well-ordered set. The direct product semigroup $\prod_{\alpha \in A} S_\alpha$ is called the *lexicographic product* of $\{S_\alpha ; \alpha \in A\}$ if the simple order $\leqq$ in $\prod_{\alpha \in A} S_\alpha$ is defined by

$(a_1, \cdots a_\alpha, \cdots) < (b_1, \cdots, b_\alpha, \cdots)$ *if and only if there exists an element* $\alpha \in A$ *such that, for every* $\gamma \in A$ *with* $\gamma < \alpha$, $a_\gamma = b_\gamma$ *and moreover that* $a_\alpha < b_\alpha$.

The purpose of this note is to give a condition in order that the lexicographic product of a well-ordered collection of ordered semigroups is an ordered semigroup.

A semigroup S is called *left* [*right*] *condensed* if, for every $s \in S$, sS [Ss] is a one-element set.

Lemma 1. *Let S be a left condensed semigroup. Then there exist a partition of S into $\{T_\lambda ; \lambda \in \Lambda\}$ and, for each $\lambda \in \Lambda$, an element $z_\lambda \in T_\lambda$ such that z_λ is a left zero of the semigroup S and that, for every $x_\lambda \in T_\lambda$, $x_\lambda S = z_\lambda$.*

Proof. Let S be a left condensed semigroup. For $a, b \in S$, we define $a \sim b$ if and only if $aS = bS$. Then the relation $\sim$ is an equivalence relation. Hence the set of equivalence classes $\{T_\lambda ; \lambda \in \Lambda\}$ forms a partition of S. By definition, for each T_λ, there corresponds an element $z_\lambda \in S$ such that $x_\lambda S = z_\lambda$ for every $x_\lambda \in T_\lambda$. Hence

$$z_\lambda S = x_\lambda S^2 = z_\lambda$$

and so z_λ is a left zero of S and moreover $z_\lambda \in T_\lambda$.

Lemma 2. *A semigroup S is left condensed and left cancellative if and only if S consists of one element.*

Proof. Let S be a left condensed and left cancellative semigroup and let $x, y \in S$. Since S is left condensed, we have $x^2 = xy$ and then, since S is left cancellative, $x = y$. Hence S consists of one element. The converse part is trivial.

Lemma 3. *A semigroup S is left condensed and right cancellative* *if and only if S is a left zero semigroup.*

Proof. Let S be a left condensed and right cancellative semigroup and let $x, y \in S$. Then $xyx=x^2$ and so $xy=x$. Hence S is a left zero semigroup. Conversely let S be a left zero semigroup. Then, for each $x \in S$, we have $xS=x$. Also if $xz=yz$, then $x=xz=yz=y$.

Lemma 4. *A semigroup S is left condensed and right condensed if and only if S^2 consists of one element.*

Proof. If S^2 consists of one element, then trivially S is left condensed and right condensed. Next suppose that S^2 contains at least two different elements, say xy and uv. Then we have either $xy \neq xv$ or $xv \neq uv$. If $xy \neq xv$, then S is not left condensed, and, if $xv \neq uv$, then S is not right condensed.

Theorem 5. *Let $\{S_\alpha; \alpha \in A\}$ be a well-ordered collection of left ordered semigroups. Then the lexicographic product $\prod_{\alpha \in A} S_\alpha$ is a left ordered semigroup if and only if it satisfies either one of the following two conditions:*

(1) *For every $\alpha \in A, S_\alpha$ is left cancellative;*

(2) *There exists an element $\beta \in A$ such that, for every $\alpha \in A$ with $\alpha < \beta, S_\alpha$ is left cancellative and, for every $\alpha \in A$ with $\beta < \alpha, S_\alpha$ is left condensed.*

Proof. We suppose that the lexicographic product $\prod_{\alpha \in A} S_\alpha$ is a left ordered semigroup. First we prove that, for every $\alpha, \gamma \in A$ with $\alpha < \gamma$, either S_α is left cancellative or S_γ is left condensed. By way of contradiction we assume that S_α is not left cancellative and S_γ is not left condensed. Then there exist $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$ such that $a_\alpha < b_\alpha$ and $c_\alpha a_\alpha = c_\alpha b_\alpha$, and there exist $p_\gamma, q_\gamma, r_\gamma \in S_\gamma$ such that $r_\gamma q_\gamma < r_\gamma p_\gamma$. For each $\delta \in A$ such that $\delta \neq \alpha$ and $\delta \neq \gamma$, we take $x_\delta \in S_\delta$ arbitrarily. Then we have

$$(x_1, \cdots, a_\alpha, \cdots, p_\gamma, \cdots) < (x_1, \cdots, b_\alpha, \cdots, q_\gamma, \cdots)$$

and

$$\begin{aligned}(x_1, \cdots, c_\alpha, \cdots, r_\gamma, \cdots)&(x_1, \cdots, a_\alpha, \cdots, p_\gamma, \cdots)\\ &= (x_1^2, \cdots, c_\alpha a_\alpha, \cdots, r_\gamma p_\gamma, \cdots)\\ &> (x_1^2, \cdots, c_\alpha b_\alpha, \cdots, r_\gamma q_\gamma, \cdots)\\ &= (x_1, \cdots, c_\alpha, \cdots, r_\gamma, \cdots)(x_1, \cdots, b_\alpha, \cdots, q_\gamma, \cdots),\end{aligned}$$

which contradicts the assumption that $\prod_{\alpha \in A} S_\alpha$ is a left ordered semigroup. Now we suppose that there exists $\beta \in A$ such that S_β is neither left cancellative nor left condensed. Then, by the result just proved, for every $\alpha \in A$ such that $\alpha < \beta, S_\alpha$ is left cancellative and, for every $\alpha \in A$ such that $\beta < \alpha, S_\alpha$ is left condensed. Hence Condition (2) is satisfied. Next we suppose that, for every $\alpha \in A, S_\alpha$ is either left cancellative or left condensed. If, for every $\alpha \in A, S_\alpha$ is left cancellative, then Condition (1) is satisfied. Now we suppose that there exists $\gamma \in A$ such that S_γ is not left cancellative. We denote by β the least element of the set of elements $\gamma \in A$ such that S_γ is not left cancellative. Then, for every $\alpha \in A$ such that $\alpha < \beta$, S_α is left cancellative. Moreover, for every $\alpha \in A$ such that $\beta < \alpha$, S_α is left condensed, since $\beta < \alpha$ and S_β is not left cancellative. Hence we have Condition (2).

Conversely we suppose that $\{S_\alpha ; \alpha \in A\}$ satisfies either Condition (1) or Condition (2). We take

$$(\cdots, a_\alpha, \cdots), (\cdots, b_\alpha, \cdots), (\cdots, c_\alpha, \cdots) \in \prod_{\alpha \in A} S_\alpha$$

with

$$(\cdots, a_\alpha, \cdots) < (\cdots, b_\alpha, \cdots).$$

Then there exists $\beta \in A$ such that $a_\beta < b_\beta$ and, for every $\alpha \in A$ with $\alpha < \beta$, $a_\alpha = b_\alpha$. First we suppose that Condition (1) is satisfied. Then S_β is left cancellative and so $c_\beta a_\beta < c_\beta b_\beta$. Moreover, for every $\alpha \in A$ with $\alpha < \beta$, $c_\alpha a_\alpha = c_\alpha b_\alpha$. Hence

$$\begin{aligned}(\cdots, c_\alpha, \cdots)(\cdots, a_\alpha, \cdots) &= (\cdots, c_\alpha a_\alpha, \cdots) \\ &< (\cdots, c_\alpha b_\alpha, \cdots) = (\cdots, c_\alpha, \cdots)(\cdots, b_\alpha, \cdots).\end{aligned}$$

Next we consider the case when Condition (2) is satisfied. Then there exists $\gamma \in A$ such that, for every $\alpha \in A$ with $\alpha < \gamma$, S_α is left cancellative and, for every $\alpha \in A$ with $\gamma < \alpha$, S_α is left condensed. If $\beta < \gamma$, then S_β is left cancellative and so, by the same way as above,

$$(\cdots, c_\alpha, \cdots)(\cdots, a_\alpha, \cdots) < (\cdots, c_\alpha, \cdots)(\cdots, b_\alpha, \cdots).$$

If $\gamma \leqq \beta$, then, for every $\alpha \in A$ with $\alpha < \beta$, $c_\alpha a_\alpha = c_\alpha b_\alpha$ since $a_\alpha = b_\alpha$. Also $c_\beta a_\beta \leqq c_\beta b_\beta$ since S_β is a left ordered semigroup, and finally, for every $\alpha \in A$ with $\beta < \alpha$, $c_\alpha a_\alpha = c_\alpha b_\alpha$ since S_α is left condensed. Hence

$$(\cdots, c_\alpha, \cdots)(\cdots, a_\alpha, \cdots) \leqq (\cdots, c_\alpha, \cdots)(\cdots, b_\alpha, \cdots).$$

Hence in both cases $\prod_{\alpha \in A} S_\alpha$ is a left ordered semigroup.

Corollary 6. *Let S_1 and S_2 be left ordered semigroups. Then the lexicographic product $S_1 \times S_2$ is a left ordered semigroup if and only if it satisfies either one of the following two conditions:*

(1) *S_1 is left cancellative;*

(2) *S_2 is left condensed.*

Theorem 7. *Let $\{S_\alpha ; \alpha \in A\}$ be a well-ordered collection of ordered semigroups. Then the lexicographic product $\prod_{\alpha \in A} S_\alpha$ is an ordered semigroup if and only if $\{S_\alpha ; \alpha \in A\}$ satisfies either one of the following conditions:*

(1) *For every $\alpha \in A$, S_α is cancellative;*

(2) *There exists an element $\beta \in A$ such that, for every $\alpha \in A$ with $\alpha < \beta$, S_α is cancellative, S_β is left cancellative, and, for every $\alpha \in A$ with $\beta < \alpha$, S_α is a right zero semigroup;*

(3) *There exists an element $\beta \in A$ such that, for every $\alpha \in A$ with $\alpha < \beta$, S_α is cancellative, S_β is right cancellative, and, for every $\alpha \in A$ with $\beta < \alpha$, S_α is a left zero semigroup;*

(4) *There exists an element $\beta \in A$ such that, for every $\alpha \in A$*

with $\alpha<\beta$, S_α *is cancellative, and, for every* $\alpha\in A$ *with* $\beta<\alpha$, S_α^2 *consists of one element*;

(5) *There exist elements* $\beta_1, \beta_2 \in A$ *with* $\beta_1<\beta_2$ *such that, for every* $\alpha\in A$ *with* $\alpha<\beta_1$, S_α *is cancellative,* S_{β_1} *is left cancellative, for every* $\alpha\in A$ *with* $\beta_1<\alpha<\beta_2$, S_α *is a right zero semigroup,* S_{β_2} *is right condensed, and, for every* $\alpha\in A$ *with* $\beta_2<\alpha$, S_α^2 *consists of one element*;

(6) *There exist elements* $\beta_1, \beta_2\in A$ *with* $\beta_1<\beta_2$ *such that, for every* $\alpha\in A$ *with* $\alpha<\beta_1$, S_α *is cancellative,* S_{β_1} *is right cancellative, for every* $\alpha\in A$ *with* $\beta_1<\alpha<\beta_2$, S_α *is a left zero semigroup,* S_{β_2} *is left condensed and, for every* $\alpha\in A$ *with* $\beta_2<\alpha$, S_α^2 *consists of one element.*

Proof. An immediate consequence of Theorem 5, Lemmas 3 and 4 and their duals.

Corollary 8. *Let* S_1 *and* S_2 *be ordered semigroups. Then the lexicographic product* $S_1\times S_2$ *is an ordered semigroup if and only if* S_1 *and* S_2 *satisfy either one of the following conditions*:

(1) S_1 *is cancellative*;

(2) S_1 *is left cancellative and* S_2 *is right condensed*;

(3) S_1 *is right cancellative and* S_2 *is left condensed*;

(4) S_2^2 *consists of one element.*

Corollary 9. *Let* $\{S_\alpha ; \alpha\in A\}$ *be a well-ordered collection of the same ordered semigroup* $S_\alpha=S$. *Then the lexicographic product* $\prod_{\alpha\in A}S_\alpha$ *is an ordered semigroup if and only if it satisfies either one of the following conditions*:

(1) *A consists of one element*;

(2) *S is cancellative*;

(3) *S is a left zero semigroup*;

(4) *S is a right zero semigroup*;

(5) S^2 *consists of one element.*

References

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