117. On the Spaces with the σ-Star Finite Open Basis

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One of the well known theorems for the metrizability is as follows: A regular T_1-space X is metrizable if and only if there exists a σ-locally finite open basis of X.

Our purpose of this paper is to study the spaces with the σ-star finite open basis.

Let us recall the definitions of terms which are used in the statement of this paper. Let X be a topological space and $\mathfrak{A}$ be a collection of subsets of X. $\mathfrak{A}$ is said to be *point finite* (resp. *point countable*) if every point of X is contained in at most finitely (resp. countably) many elements of $\mathfrak{A}$. $\mathfrak{A}$ is *locally finite* (resp. *locally countable*) if every point of X has a neighborhood which intersects only finitely (resp. countably) many elements of $\mathfrak{A}$. $\mathfrak{A}$ is *star finite* (resp. *star countable*) if every element of $\mathfrak{A}$ intersects only finitely (resp. countably) many elements of $\mathfrak{A}$. A space X is said to be *strongly paracompact* if every open covering of X has a star finite open covering of X as a refinement. *A σ-star finite open basis* is an open basis which is the union of countably many star finite open coverings.

Finally, to state our results we need the next notation. Let $\{U_x \mid x \in X\}$ be a collection of subsets of X with the index set X, then its collection is symmetric if "$y \in U_x$" is equivalent to "$x \in U_y$".

We assume that all the spaces in this paper are T_1-spaces and for a symmetric collection $\{U_x \mid x \in X\}$, U_x contains x for every point $x \in X$.

As is well known, not every metric space has a σ-star finite basis (see Yu. M. Smirnov [5]). The existence of a σ-star finite open basis is not sufficient for a metric space to be strongly paracompact (see J. Nagata [4, p. 201]), but clearly, a strong paracompactness or a local compactness is sufficient for a metric space to be with the σ-star finite open basis, and furthermore it is known that a metric space X has a σ-star finite open basis if and only if X is homeomorphic to a subspace of a topological product $N(\Omega)$[1] $\times I^w$ for suitable Ω (see J. Nagata [4, p. 201] or [3]).

1) $N(\Omega)$ is the *generalized Baire's zero dimensional space* with respect to Ω, that is $N(\Omega)$ is the set of all sequences $(\alpha_1, \alpha_2, \cdots)$ of elements $\alpha_i \in \Omega$. The distance between two distinct points $\alpha=(\alpha_1, \alpha_2, \cdots)$ and $\beta=(\beta_1, \beta_2 \cdots)$ of $N(\Omega)$ are defined by

$$\rho(\alpha, \beta)=\frac{1}{\min\{k \mid \alpha_k \neq \beta_k\}}.$$

At first we begin by proving the following lemma.

Lemma. *Let $\{U_\alpha \mid \alpha \in A\}$ be a locally finite open covering of a normal space X, then there exists a closed covering $\{F_\alpha \mid \alpha \in A\}$ of X such that*

(i) $F_\alpha \subset U_\alpha$ *for every* $\alpha \in A$,

and

(ii) *if* $\bigcap_{\alpha \in A'} U_\alpha - \bigcup_{\alpha \notin A'} U_\alpha$ *is not empty, then* $\bigcap_{\alpha \in A'} F_\alpha - \bigcup_{\alpha \notin A'} F_\alpha$ *is not empty for every* $A' \subset A$.[2]

Proof. Let $[A]=\{A' \mid A' \subset A, \bigcap_{\alpha \in A'} U_\alpha - \bigcup_{\alpha \notin A'} U_\alpha \neq \emptyset$ and $x(A')$ be an arbitrarily fixed point of $\bigcap_{\alpha \in A'} U_\alpha - \bigcup_{\alpha \notin A'} U_\alpha$ for each $A' \in [A]$. Then, $\{\{x(A')\} \mid A' \in [A]\}$ is a locally finite closed collection (see M. Katetov [1, Theorem 1-1]).

On the other hand, there exists a closed covering $\{F'_\alpha \mid \alpha \in A\}$ of X such that $F'_\alpha \subset U_\alpha$ for every $\alpha \in A$ (see M. Katetov [1, Theorem 1-2]). If we let

$$F_\alpha = F'_\alpha \cup \{x(A') \mid \alpha \in A' \in [A]\} \qquad \text{for each } \alpha \in A,$$

then it is clear that $\{F_\alpha \mid \alpha \in A\}$ is a closed covering satisfying the properties (i) and (ii) of the lemma.

Theorem 1. *In a regular space X, the following properties are equivalent:*

(1) *There exists a σ-star finite open basis.*

(2) *There exists a basis which is the union of countably many symmetric star finite open coverings of X.*

(3) *There exists a basis which is the union of countably many symmetric locally finite open coverings of X.*

(4) *There exists a basis which is the union of countably many symmetric point finite open coverings of X.*

Proof. (1) *implies* (2). Let $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}_n$ be a σ-star finite open basis of X where $\mathfrak{U}_n = \{U_\alpha \mid \alpha \in A_n\}$ is a star finite open covering of X for $n=1, 2, \cdots$. From the above lemma, for each n, we get the closed covering $\mathfrak{F}_n = \{F_\alpha \mid \alpha \in A_n\}$ such that

(i) $F_\alpha \subset U_\alpha$ for every $\alpha \in A_n$,

and

(ii) if $\bigcap_{\alpha \in A} U_\alpha - \bigcup_{\alpha \notin A} U_\alpha$ is nonempty, then $\bigcap_{\alpha \in A} F_\alpha - \bigcup_{\alpha \notin A} F_\alpha$ is nonempty for every $A \subset A_n$.

If, for each n, we put $\mathfrak{H}_n = \bigwedge_{\alpha \in A_n} \{U_\alpha, X - F_\alpha\}$,[3] then $\mathfrak{H}_n$ will be a star

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2) This property is stronger than the property such that $\bigcap_{\alpha \in A'} U_\alpha \neq \emptyset$ is equivalent to $\bigcap_{\alpha \in A'} F_\alpha \neq \emptyset$ for every $A' \subset A$, and in the latter case, this lemma is well known.

3) $\bigwedge_{\alpha \in A_n} \{U_\alpha, X - F_\alpha\}$ is the collection $\{(\bigcap_{\alpha \in A'} U_\alpha) \cap (\bigcap_{\alpha \notin A'} (X - F_\alpha)) \mid A' \subset A_n\}$.

finite open covering of X. Really it is trivial from the star finiteness of $\mathfrak{U}_n$ that $\mathfrak{H}_n$ is an open covering of X. Let H be an arbitrary element of $\mathfrak{H}_n$, that is,

$$H=(\bigcap_{\alpha\in A} U_\alpha)\cap(X-\bigcup_{\alpha\notin A} F_\alpha) \qquad \text{for some } A\subset A_n,$$

α_0 be an arbitrarily fixed element of A and $A'=\{\alpha\in A_n \mid U_{\alpha_0}\cap U_\alpha\neq\emptyset\}$, then A' is finite. If $H\cap H'\neq\emptyset$ for $H'=(\bigcap_{\alpha\in B} U_\alpha)\cap(X-\bigcup_{\alpha\notin B} F_\alpha)\in\mathfrak{H}_n$, then $(\bigcap_{\alpha\in A} U_\alpha)\cap(\bigcap_{\alpha\in B} U_\alpha)\neq\emptyset$ and therefore $B\subset A'$. Consequently H intersects only finitely many elements of $\mathfrak{H}_n$.

Furthermore we shall prove that $\{\mathrm{st}(x,\mathfrak{G})^{4)}\mid x\in X\}$ is star finite for every star finite collection $\mathfrak{G}=\{G_\lambda\mid\lambda\in\Lambda\}$ of subsets of X. For this purpose, it is sufficient to show that $\mathfrak{G}'=\{\bigcup_{i=1}^n G_{\lambda_i}\mid \bigcap_{i=1}^n G_{\lambda_i}\neq\emptyset, G_{\lambda_i}\in\mathfrak{G}$ for $i=1,2,\cdots,n; n=1,2\cdots\}$ is star finite. Let $\Lambda_\lambda=\{\lambda'\in\Lambda\mid G_\lambda\cap G_{\lambda'}\neq\emptyset\}$ for each $\lambda\in\Lambda$, then Λ_λ is finite. If $G'=\bigcup_{i=1}^n G_{\lambda_i}$ and $G''=\bigcup_{i=1}^m G_{\mu_i}$ are elements of $\mathfrak{G}'$ where $\bigcap_{i=1}^n G_{\lambda_i}\neq\emptyset$ and $\bigcap_{i=1}^m G_{\mu_i}\neq\emptyset$, and $G'\cap G''$ is nonempty, then

$$\{\mu_1,\mu_2,\cdots,\mu_m\}\subset\bigcup\{\Lambda_\mu\mid\mu\in\bigcup\{\Lambda_\lambda\mid\lambda\in\bigcup_{i=1}^n\Lambda_{\lambda_i}\}\} \quad \text{(this set is finite)},$$

therefore $\mathfrak{G}'$ is a star finite collection of subsets of X, and hence $\mathfrak{V}_n=\{\mathrm{st}(x,\mathfrak{H}_n)\mid x\in X\}$ is a star finite open covering of X.

Lastly we will show that $\mathfrak{V}=\bigcup_{n=1}^\infty\mathfrak{V}_n$ is a basis of X and $\mathfrak{V}_n$ is a symmetric star finite open covering of X. From the above discussion, $\mathfrak{V}_n$ being a symmetric star finite open covering of X is trivial for each $n=1,2,\cdots$.

In order to prove that $\mathfrak{V}$ is a basis of X, let x be any element of G for any open set $G\subset X$. From the fact that $\mathfrak{U}=\bigcup_{n=1}^\infty\mathfrak{U}_n$ is a basis of X, there exists a positive integer n_0 and, $\alpha_0\in A_{n_0}$ such that $x\in U_{\alpha_0}\subset G$. If $A=\{\alpha\in A_{n_0}\mid x\in U_\alpha\}$, then α_0 is an element of finite set A and $x\in\bigcap_{\alpha\in A}U_\alpha-\bigcup_{\alpha\notin A}U_\alpha$. Then, from the property (ii) of $\mathfrak{F}_{n_0}$, $\bigcap_{\alpha\in A}F_\alpha-\bigcup_{\alpha\notin A}F_\alpha$ is not empty, and let y be any element of it. If we let $H=(\bigcap_{\alpha\in A}U_\alpha)\cap(X-\bigcup_{\alpha\notin A}F_\alpha)$, then $x,y\in H\in\mathfrak{H}_{n_0}$, i.e., $x\in\mathrm{st}(y,\mathfrak{H}_{n_0})\in\mathfrak{V}_{n_0}$.

Our next step is the fact that $\mathrm{st}(y,\mathfrak{H}_{n_0})$ is contained in U_{α_0}. Let $H=(\bigcap_{\alpha\in B}U_\alpha)\cap(X-\bigcup_{\alpha\notin B}F_\alpha)$ be any element of $\mathfrak{H}_{n_0}$ which contains the point y. From $y\in\bigcap_{\alpha\in A}F_\alpha$ and $\alpha_0\in A$, we get $y\in F_{\alpha_0}$ and hence $\alpha_0\in B$,

4) For the collection $\mathfrak{G}$ of subsets of X and the subset A of X, $\mathrm{st}(A,\mathfrak{G})$ is the union of all elements of $\mathfrak{G}$ which intersect A.

because y is not contained in $\bigcup_{\alpha \notin B} F_\alpha$. Therefore $H \subset \bigcap_{\alpha \in B} U_\alpha \subset U_{\alpha_0}$, that is,

$$x \in \mathrm{st}(y, \mathfrak{H}_{n_0}) \subset U_{\alpha_0} \subset G,$$

and hence $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$ is a basis of X such that $\mathfrak{V}_n$ is a symmetric star finite open covering of X for $n=1, 2, \cdots$. It completes the proof.

(2) *implies* (3), (3) *implies* (4): It is trivial.

(4) *implies* (2). The symmetric point finiteness is the symmetric star finiteness (see Y. Yasui [7]).

(2) *implies* (1). It is trivial.

It completes the proof of Theorem 1.

In J. Nagata [3, Remark], a regular space X has a star finite basis if and only if X has a σ-star countable basis. Then, we will get the following theorem:

Theorem 2. *In a regular space X, the following properties are equivalent*:

(1) *There exists a σ-star finite basis.*

(2) *There exists a basis which is the union of countably many symmetric star countable open coverings of X.*

(3) *There exists a basis which is the union of countably many symmetric locally countable open coverings of X.*

(4) *There exists a basis which is the union of countably many symmetric point countable open coverings of X.*

(5) *There exists a σ-star countable basis, that is, a basis which is the union of countably many star countable open coverings of X.*

Proof. (5) *implies* (1). We assume that there exists an open basis $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}_n$ such that $\mathfrak{U}_n = \{U_\alpha^n \mid \alpha \in A_n\}$ is a star countable open covering of X, then from the star countability of $\mathfrak{U}_n$, we can get the decomposition $\bigcup_{\lambda \in \Lambda_n} \Gamma_\lambda^n$ of A_n such that α, β being the same class Γ_λ^n is equivalent to $U_\beta^n \subset \bigcup_{i=1}^{\infty} \mathrm{st}^i(U_\alpha^n, \mathfrak{U}_n)$[5] (or $U_\alpha^n \subset \bigcup_{i=1}^{\infty} \mathrm{st}^i(U_\beta^n, \mathfrak{U}_n)$). If we let $G_\lambda^n = \cup \{U_\alpha^n \mid \alpha \in \Gamma_\lambda^n\}$, then it is easily seen that $\{G_\lambda^n \mid \lambda \in \Lambda_n\}$ is a mutually disjoint open covering of X for $n=1, 2, \cdots$. From the star countability of $\mathfrak{U}_n$, we may put $\Gamma_\lambda^n = \{\alpha_i^{n\lambda} \mid i=1, 2, \cdots\}$.

For every positive integer n and i, we let $\mathfrak{V}^{n,i} = \{U_{\alpha_i^{n\lambda}}^n \mid \lambda \in \Lambda_n\}$. Then, since $U_{\alpha_i^{n\lambda}}^n \subset G_\lambda^n$ for each $\lambda \in \Lambda_n$, and $\{G_\lambda^n \mid \lambda \in \Lambda_n\}$ is a discrete covering of X, it is trivial for $\mathfrak{V}^{n,i}$ to be a discrete open collection of subsets of X. On the other hand, $\bigcup_{i=1}^{\infty} \mathfrak{V}^{n,i} = \mathfrak{U}_n$ is easily seen, and therefore $\bigcup_{n,i=1}^{\infty} \mathfrak{V}^{n,i} = \bigcup_{n=1}^{\infty} \mathfrak{U}_n = \mathfrak{U}$ is a basis of X. Furthermore, if let

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5) for $i>1$, $\mathrm{st}(U_{\alpha_i}^n, \mathfrak{U}_n) = \mathrm{St}(\mathrm{st}^{i-1}(U_\alpha^n, \mathfrak{U}_n), \mathfrak{U}_n)$.

$\mathfrak{H}^{n,i}=\mathfrak{V}^{n,i}\cup\{G_\lambda^n\mid\lambda\in\Lambda_n\}$, then $\mathfrak{H}^{n,i}$ is an open covering of X, because $\{G_\lambda^n\mid\lambda\in\Lambda_n\}$ is a covering of X, and hence $\mathfrak{H}^{n,i}$ is a star finite open covering (really each element of $\mathfrak{H}^{n,i}$ intersects at most one other element of $\mathfrak{H}^{n,i}$).

From the above, $\mathfrak{H}=\bigcup_{n,i=1}^{\infty}\mathfrak{H}^{n,i}\supset\bigcup_{n,i=1}^{\infty}\mathfrak{V}^{n,i}=\bigcup_{n=1}^{\infty}\mathfrak{U}_n=\mathfrak{U}$ is a basis and of course $\mathfrak{H}$ is a basis such that $\mathfrak{H}^{n,i}$ is a star finite open covering of X, that is, $\mathfrak{H}$ is a σ-star finite open basis of X.

(1) *implies* (5). It is trivial.

(1), (2), (3) *and* (4) *are equivalent.* It is trivial from the facts that (1) implies (2) of Theorem 1, (2) of Theorem 1 implies (2), (2) implies (3), (3) implies (4), (4) implies (2) (see Y. Yasui [7]), (2) implies (5) and (5) implies (1).

It completes the proof of Theorem 2.

Remark 1. The σ-star finite open basis is the basis $\mathfrak{U}$ which is the union of countably many star finite open coverings $\mathfrak{U}_n$ of X. In this definition, we can not give the star finite and locally finite collection of open sets of X instead of each $\mathfrak{U}_n$ being the star finite open covering of X, that is, there exists a space X with the basis which is the union of countably many star finite and locally finite open collection of X, but X has not the σ-star finite open basis. Really, let X be the Euclidean plane set and ρ be a usual metric on X.

We shall define the following other metric d on X (see Yu. M. Smirnov [5], or Y. Yasui [6]):

$$d(x,y)=\begin{cases}\rho(x,0)+\rho(y,0) & \text{if argument } x\not\equiv\text{argument } y \pmod{\pi}\\ \rho(x,y) & \text{if argument } x\equiv\text{argument } y \pmod{\pi},\end{cases}$$

Where 0 is the original point of X.

Then it is easily seen that this metric space (X,d) has an open basis $\mathfrak{U}=\bigcup_{n=1}^{\infty}\mathfrak{U}_n$ such that each $\mathfrak{U}_n$ is a star finite and locally finite collection of open sets of X. But X will not be with a σ-star finite open basis.

If X has a σ-star finite open basis $\mathfrak{U}=\bigcup_{n=1}^{\infty}\mathfrak{U}_n$ where each $\mathfrak{U}_n$ is a star finite open covering of X, then each $\mathfrak{U}_n$ is a countable collection because of connectedness of X and hence $\mathfrak{U}$ is a countable open basis. But it is clear that X is not separable.

Remark 2. Separable metric spaces (or in general, locally separable metric spaces) are the spaces with the σ-star finite open basis, and the converse is not true. Really, an uncountable space with the discrete topology has this property.

Remark 3. Not every strongly paracompact spaces are the spaces with the σ-star finite open basis (see J. Nagata [4, p. 201]) and, not the spaces with the σ-star finite open basis are the strongly paracompact spaces (see Example of Remark 1).

References

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