115. *Boundary Behaviour of Functions Harmonic in the Unit Ball*

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1. The main purpose of this note is to prove Meier's theorem ([5], Satz 5, cf. [2], p. 154) in a real-harmonic form in the open unit ball U whose centre is the origin O in the Euclidean space R^3.

We begin with definitions of cluster sets following the planar cases (cf. [2], [6]). The two-point compactification $R^1\cup\{-\infty, +\infty\}$ of the real number system R^1 is denoted by R^*. Let Ω be a domain in R^3, Q be a point of the boundary $\partial\Omega$ and $\mathcal{G}$ be a subset of Ω whose closure $\overline{\mathcal{G}}$ in R^3 contains Q. Let $f(P)$ be a real-valued function in Ω. Then, the cluster set of f at Q along $\mathcal{G}$ is defined by

$$C_{\mathcal{G}}(f, Q)=\bigcap_{r>0}\overline{f(\delta_r\cap\mathcal{G})},$$

where δ_r is the open ball $\{P; \overline{PQ}<r\}$ and the closure is taken in R^*. By a *cone* $\Delta=\Delta(Q, \varphi, h)$ (in Ω) at Q we mean an open circular cone in Ω with vertex Q, axis along a straight line through Q, generating angle (=one half of the opening angle) $\varphi, 0<\varphi<\pi/2$, and altitude h. A *segment* X (in Ω) at Q is an open rectilinear segment X in Ω terminating at Q. The cluster sets corresponding to $\mathcal{G}=\Omega, \Delta$ and X will be denoted by $C_\Omega(f, Q), C_\Delta(f, Q)$ and $C_X(f, Q)$ respectively; these sets are non-empty and closed in R^* and in the case where f is continuous, they are, except possibly for $C_\Omega(f, Q)$, connected, i.e., of a form of "interval" $[a, b]$, $a, b\in R^*$.

A point $Q\in\partial\Omega$ is called a *Plessner point* of f if for any cone Δ at Q, $C_\Delta(f, Q)=R^*$. A *Fatou point* $Q\in\partial\Omega$ of f is a point at which $\bigcup_\Delta C_\Delta(f, Q)$ consists of a single point of R^*; here, Δ ranges over all cones at Q. A point $Q\in\partial\Omega$ is called a *Meier point* of f if $\bigcap_X C_X(f, Q)$ $=C_\Omega(f, Q)\neq R^*$, where X ranges over all segments at Q. The totality of Plessner (Fatou, Meier, resp.) points of f will be denoted by $I(f, \Omega)$ $(F(f, \Omega), M(f, \Omega)$, resp.).

Our main theorem is stated in the case where Ω is the ball.

Theorem 1. *Let f be harmonic in the ball $U=\{P; \overline{OP}<1\}$. Then*

$$\partial U\setminus\{I(f, U)\cup M(f, U)\}$$

is of first category in Baire's sense on the unit sphere ∂U.

Meier's theorem is usually called "topological analogue of

Plessner's theorem". For the reader's convenience we shall prove the harmonic Plessner's theorem in its full form (cf. [2], p. 147 for the meromorphic form).

Theorem 2. *Let f be harmonic in the ball U. Then*

$$\partial U \setminus \{I(f, U) \cup F(f, U)\}$$

is of Lebesgue measure zero on U.

2. Let f be harmonic in U. Let $\Delta_1(P_o), \cdots, \Delta_j(P_o), \cdots$ be a countable number of cones in U at $P_o=(1,0,0)$ such that any cone $\Delta(P_o)$ at P_o contains at least one $\Delta_j(P_o)$. Let $\Delta_j(Q)$ be the cone at $Q \in \partial U$ obtained by rotation of $\Delta_j(P_o)$ around O $(j=1,2,\cdots)$. Let $k_1, \cdots, k_\nu, \cdots$ be the totality of rational numbers. Then for any point Q of the set $E=\partial U \setminus I(f, U)$ we may find one $\Delta_j(Q)$ and k_ν such that $\overline{f(\Delta_j(Q))}$ lies on the right-hand-side (simply, "on the right") or on the left-hand-side ("on the left") of k_ν. We denote by $E_{j,\nu,r}$ ($E_{j,\nu,l}$, resp.) the set of points $Q \in E$ at which $\overline{f(\Delta_j(Q))}$ lies on the right (left, resp.) of k_ν; these sets are closed on ∂U. We then obtain the following decomposition of E.

(1) $$E=\bigcup_{j,\nu} \{E_{j,\nu,r} \cup E_{j,\nu,l}\}.$$

We first give a sketch of

Proof of Theorem 2. Set $E^*=E \setminus F(f, U)$ and decompose

(2) $$E^*=\bigcup_{j,\nu} \{E^*_{j,\nu,r} \cup E^*_{j,\nu,l}\},$$

where $E^*_{j,\nu,\alpha}=E^* \cap E_{j,\nu,\alpha}$, $\alpha=r, l$. Since $F(f, U)$ is measurable as in the plane case (cf. [7], p. 219, the foot-note), the measurability of $E^*_{j,\nu,\alpha}$ follows from:

$$E^*_{j,\nu,\alpha}=E_{j,\nu,\alpha} \setminus (E_{j,\nu,\alpha} \setminus E^*_{j,\nu,\alpha}) \quad \text{and} \quad E_{j,\nu,\alpha} \setminus E^*_{j,\nu,\alpha}=E_{j,\nu,\alpha} \cap F(f, U)$$
$$\text{for } j, \nu=1,2,\cdots; \alpha=r, l.$$

We shall prove that all sets $E^*_{j,\nu,\alpha}$ are of measure zero. Assume otherwise. Then, we have one $E^*_{j,\nu,\alpha}$, for example, $E^*_{j,\nu,r}$ of positive measure. We can now apply Carleson-Hunt-Wheeden's theorem (cf. [3], Theorems at p. 308 and p. 321) to $f-k_\nu$ on U. Then, the points of $E^*_{j,\nu,r}$ are, except for a set of measure zero, Fatou points. This contradicts $E^*_{j,\nu,r} \subset E^*=E \setminus F(f, U)$. Q.E.D.

We now discuss Theorem 1. Let G be a subdomain of the sphere ∂U and let $\Delta(Q_o)$ be a cone in U at $Q_o \in G$. Let $\Delta(Q)$ be the cone at $Q \in G$ obtained by rotation of $\Delta(Q_o)$ around O. We first consider in the domain $\Omega=\bigcup_{Q \in G} \Delta(Q)$.

Theorem 3. *Let Ω be as defined above and let f be non-negative and harmonic in Ω. Then, $G \setminus M(f, \Omega)$ is of first Baire category on G.*

Lemma 1. *Let Ω be as in Theorem 3 and let g be an arbitrary real-valued function in Ω. Then, $G \setminus J(g, \Omega)$ is of first category on G, where $J(g, \Omega)$ is the set of points $Q \in \partial\Omega$ at which $C_\Delta(g, Q)=C_\Omega(g, Q)$ holds for any cone $\Delta \subset \Omega$ at Q.*

Proof of Lemma 1. Let $\{G_n\}$ be a sequence of subdomains of G on ∂U such that $\bar{G}_n \subset G_{n+1} \subset G$ for $n=1,2,\cdots$, and $G=\bigcup_n G_n$. Then we have: $G_n \backslash J(g,\Omega)$ is of first category on G_n and hence on G. The proof of this follows the same line as in the proof of Theorem 6 by Collingwood [1]. The lemma now follows from

$$G \backslash J(g,\Omega)=\bigcup_n \{G_n \backslash J(g,\Omega)\}.$$

Lemma 2 ([4], p. 262). *Let $u(P)$ be non-negative and harmonic on the closed ball $\{P; \overline{OP} \leqq a\}$. Then, putting $\rho=\overline{OP}$, we have*

(3) $$a(a-\rho)(a+\rho)^{-2}u(O) \leqq u(P) \leqq a(a+\rho)(a-\rho)^{-2}u(O).$$

Proof of Theorem 3. Let $Q \in G \backslash M(f,\Omega)$. Then we have a segment X at Q such that $C_X(f,Q) \neq C_\Omega(f,Q)$. Since f is positive we can choose a positive number $\alpha \in C_\Omega(f,Q) \backslash C_X(f,Q)$. As $C_X(f,Q)$ is connected, this must lie on the right or on the left of α. First we consider the "right" case with an additional condition

(4) $$C_X(f,Q) \cap R^1 \neq \emptyset \text{ (non-empty)}.$$

Let $\beta=(1/4)\operatorname{dis}\{\alpha, C_X(f,Q) \cap R^1\}$ (>0), where "dis" means the usual distance in R^1. By compactness of $C_X(f,Q)$ we may find a subsegment X_1 of X terminating at Q such that $\overline{f(X_1)}$ lies on the right of $\alpha+2\beta$. Furthermore, by the property of Ω, we may assume that there exists a cone $\Delta_1=\Delta_1(Q,\varphi,h)$ at Q lying in Ω and whose axis contains X_1 and $h>$(the length of X_1). Let $\mu>0$ be a constant such that

(5) $$(\alpha+\beta)(\alpha+2\beta)^{-1}<(1-\mu)(1+\mu)^{-2}.$$

Let $P_1 \in X_1$ and let $\bar{\delta}(P_1)$ be the closed ball with centre P_1 and radius $a(P_1)=(1/2)\overline{QP}_1 \sin\varphi$. Let $\delta^*(P_1)$ be the open ball with centre P_1 and radius $\mu a(P_1)$. We then apply Lemma 2 (the left-hand-side of (3)) to f on $\bar{\delta}(P_1)$. Then for $P \in \delta^*(P_1) \subset \bar{\delta}(P_1)$ we have

(6) $$f(P)>(1-\mu)(1+\mu)^{-2}f(P_1)>\alpha+\beta$$

by (5) and $f(P_1) \geqq \alpha+2\beta$. Now, as $X_1 \ni P_1 \to Q$, the balls $\delta^*(P_1)$ *cover* a cone Δ at Q and hence by (6) we know that $\overline{f(\Delta)}$ lies on the right of $\alpha+\beta$. This means that $\alpha \notin C_\Delta(f,Q)$ or

(7) $$C_\Delta(f,Q) \neq C_\Omega(f,Q).$$

In the case where the set on the left-hand-side of (4) is empty, we have (7) by Lemma 2, since $f(P) \to +\infty$ as $X \ni P \to Q$. In the "left" case the proof is similar.

Combined with Lemma 1, (7) proves our Theorem 3. Q.E.D.

Remark. The above method of using Harnack's inequality can be used in the proof of the *three-dimensional extension* of Tsuji's theorem (Theorem IV. 20., p. 152 of [8]).

Proof of Theorem 1. We consider the decomposition (1) of the set $E=\partial U \backslash I(f,U)$. If E is of first category, we have nothing to prove. We assume that E is of second category. Then at least one of $E_{j,\nu,\alpha}$,

$j,\ \nu=1,2,\cdots;\ \alpha=r,l$, is of second category. Let $A=E_{j,\nu,r}$, for example, be such one and let B be the boundary of the closure $\bar{A}$ of A in ∂U, so that $\bar{A}\backslash B$ consists of countably many open components G_μ on ∂U. Let $G=G_\mu$ be one of them. Then, $A\cap G$ is dense in G and hence we obtain a domain $\Omega=\bigcup_{Q\in G}\Delta_j(Q)$ such that $\overline{f(\Omega)}$ lies on the right of k_ν. By Theorem 3 for $h=f-k_\nu$, we know that

(8) $$G\backslash M(h,\Omega)=G\backslash M(f,\Omega)$$

is of first category on G and hence on ∂U. On the other hand, by the property of the domain Ω, we have $M(f,\Omega)\cap G=M(f,U)\cap G$, and hence by (8), the set $G\backslash M(f,U)$ is of first category on ∂U. Since $\bar{A}=B\cup\{\bigcup_\mu G_\mu\}$, B being nowhere dense in ∂U, we have the theorem by (1). Q.E.D.

Remark. The conformal map from U onto the half-space $H=\{P=(x,y,z);\ z>0\}$ (composed map of a translation and an inversion) and the Kelvin transformation ([4], p. 232) allow us to assert the three theorems posed above in H.

References

[1] E. F. Collingwood: Cluster set theorems for arbitrary functions with applications to function theory. Ann. Acad. Sci. Fenn., No. 336/8, 1-15 (1963).

[2] E. F. Collingwood and A. J. Lohwater: The Theory of Cluster Sets. Cambridge Univ. Press, Cambridge (1966).

[3] R. A. Hunt and R. L. Wheeden: On the boundary values of harmonic functions. Trans. Amer. Math. Soc., **132**, 307-322 (1968).

[4] O. D. Kellogg: Foundations of Potential Theory. Springer-Verlag, Berlin-Heidelberg-New York (1967).

[5] K. E. Meier: Über die Randwerte der meromorphen Funktionen. Math. Ann., **142**, 328-344 (1961).

[6] K. Noshiro: Cluster Sets. Springer-Verlag, Berlin-Göttingen-Heidelberg (1960).

[7] I. I. Priwalow: Randeigenschaften analytischer Funktionen. Deutscher Verlag der Wiss., Berlin (1956).

[8] M. Tsuji: Potential Theory in Modern Function Theory. Maruzen Co., Tokyo (1959).