156. *Approximation of Obstacles by High Potentials; Convergence of Scattering Waves*

By Reiji KONNO
Senshu University

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1. Introduction and summary. In this paper it is shown that the solutions (the outgoing scattering waves) of the Dirichlet zero boundary value problem for the equation

$$(-\Delta+q(x))\varphi=|k|^2\varphi \tag{1.1}$$

in an exterior domain in 3-space is approximated in a certain sense by the solutions φ_n of the equations in the whole space

$$(-\Delta+q(x)+n\chi_K(x))\varphi_n=|k|^2\varphi_n, \qquad n=1,2,\cdots. \tag{1.2}$$

Here K is an obstacle and χ_K denotes the characteristic function of K.

The same problem for discrete eigenvalues was considered by K. Ōeda [5] and he showed the convergence of negative eigenvalues and eigenfunctions. There are some other convergence problems of this type. For instance, P. Werner [7] proved that the scattering waves for the exterior Neumann problem can be approximated by the waves in the whole space when the damping factor is made large in the obstacle.

Let K be a compact set $\subset R^3$. We suppose that ∂K, the boundary of K, is a closed surface of class C^2, and that the complement of K is connected. The potential function $q(x)$ is assumed to be a real-valued, Hölder continuous function of $x\in R^3$, satisfying the inequality $|q(x)|\leqq C|x|^{-3-\delta}$ for $|x|\geqq R_0$ with positive constants C,δ and R_0. The non-zero 3-vector k is fixed throughout this paper.

In what follows, we denote by $H^m(\Omega)$ the Sobolev space of order m over a domain Ω, by $\| \ \|_{m,\Omega}$ the norm in $H^m(\Omega)$ and by $H^m_{\mathrm{loc}}(\Omega)$ the local space. Furthermore, let us denote by $H^m_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.

Problem (I). *Find a function $\varphi(x)$ on R^3-K which satisfies the following conditions*: i) $\zeta\varphi\in H^2(R^3-K)\cap H^1_0(R^3-K)$ *where $\zeta(x)$ is an arbitrary function in $C^\infty(R^3-K)$ with $\zeta(x)\equiv 1$ near ∂K and $\zeta(x)\equiv 0$ for large $|x|$*[1]; ii) $\varphi(x)$ *satisfies* (1.1) *in the sense of distributions*; iii) $v(x)\equiv\varphi(x)-e^{ik\cdot x}$ *satisfies the radiation condition at infinity*

$$v(x)=O(|x|^{-1}),\quad (\partial/\partial|x|-i|k|)v(x)=o(|x|^{-1})\qquad (|x|\to\infty) \tag{1.3}$$

uniformly with respect to every direction.

1) Under these assumptions, however, φ becomes the classical solution. That is, φ is twice continuously differentiable in R^3-K and continuous up to the boundary.

Problem (II). *Find a function* $\varphi_n(x)$ *on* R^3 *which satisfies the following conditions*; i) $\varphi_n \in H^2_{\mathrm{loc}}(R^3)$; ii) $\varphi_n(x)$ *satisfies* (1.2) *in the sense of distributions*; iii) $v_n(x) \equiv \varphi_n(x) - e^{ik\cdot x}$ *satisfies the radiation condition* (1.3).

The existence and the uniqueness of the solutions of Problems (I) and (II) have been proved under weaker assumptions. (Cf. Ikebe [1], [2], Werner [6].) Though the main purpose of the present work is to prove the convergence of φ_n to φ, it gives another proof of the existence of φ. Our proof is essentially based on the uniqueness of φ (Lemma 3), which was obtained as a consequence of Kato's theorem on the growth properties of solutions ([3]).

Theorem. *There exists a unique solution* φ *of* (I), *and* $\varphi_n \to \varphi$ *as* $n \to \infty$ *in the following sense*: i) *in* $H^2_{\mathrm{loc}}(R^3 - K)$; ii) *in* $L_{2,\mathrm{loc}}(R^3)$ *if* $\varphi(x)$ *is extended to the whole space by setting* $\varphi(x)=0$ *in* K; iii) *uniformly in* $R^3 - D$ *where* D *is an arbitrary bounded open set including* K.

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2. Proof of the theorem. In what follows, we take and fix a set D having the properties described in the theorem. Let $\psi_n = c_n\varphi_n = c_n e^{ik\cdot x} + w_n$ $(c_n > 0)$. We put a new normalization condition on ψ_n,

$$c_n + \| w_n \|_{0,D} + \sup_{x\in R^3 - D} |w_n(x)| = 1 \tag{2.1}$$

where $\| \ \|_{0,D}$ means the L_2-norm over D. (Note that $\varphi_n(x)$ is necessarily bounded in the whole space.)

Lemma 1. *There exists a subsequence of* ψ_n *which converges to a* $\psi \in H^1_{\mathrm{loc}}(R^3)$ *in the following sense*: i) *in* $L_2(D)$; ii) *in* $H^2_{\mathrm{loc}}(R^3 - K)$; iii) *uniformly in* $R^3 - D$.

Proof. Let Ω, Ω' be arbitrary bounded open sets such that $\Omega \subset\subset \Omega' \subset R^3 - K$. Here $A \subset\subset B$ means that A and B are open, $\bar{A}$, the closure of A, is compact and $\bar{A} \subset B$. Since $\Delta\psi_n = (q - |k|^2)\psi_n$ in Ω', it follows from the interior estimate for Δ that

$$\| \psi_n \|_{2,\Omega} \leqq \text{const} \cdot \| \psi_n \|_{0,\Omega'}. \tag{2.2}$$

Thus, noting that $\| \psi_n \|_{0,\Omega'}$ is bounded by (2.1), we have $\| \psi_n \|_{2,\Omega} \leqq$ const. and hence, in particular, $\| \nabla \psi_n \|_{0,\Omega} \leqq$ const.

Now we consider a selfadjoint operator $A_n = -\Delta + q + n\chi_K$ in $L_2(R^3)$ with domain $D(A) = H^2(R^3)$. Since A_n is bounded below uniformly with respect to n, one has $\| (A_n + t)^{-1} \| \leqq M$, $n = 1, 2, \cdots$, for sufficiently large constants t and M. Let $\alpha \in C_0^\infty(R^3)$ be such that $\alpha(x) \equiv 1$ on D. Then $u_n \equiv (A_n + t)(\alpha\psi_n) = (|k|^2 + t)\alpha\psi_n - 2\nabla\alpha\cdot\nabla\psi_n - (\Delta\alpha)\psi_n$, $n = 1, 2, \cdots$, form a bounded sequence in $L_2(R^3)$ because the support of $\nabla\alpha\cdot\nabla\psi_n$ is compact and $\subset R^3 - K$. Hence, from

$$M \| u_n \|^2 \geqq \int_{R^3} [(A_n + t)^{-1} u_n] \bar{u}_n dx = \int_{R^3} (\alpha\psi_n)[\overline{(A_n + t)(\alpha\psi_n)}] dx \tag{2.3}$$

$$=t\|\alpha\psi_n\|_{0,R^3}^2+n\|\chi_K\alpha\psi_n\|_{0,R^3}^2+\|\nabla(\alpha\psi_n)\|_{0,R^3}^2$$
$$+\int_{R^3}(q\alpha\psi_n)\overline{\alpha\psi_n}dx,$$

we obtain $\|\nabla(\alpha\psi_n)\|_{0,R^3}\leqq$const. According to Rellich's theorem we can find a subsequence of ψ_n converging in the sense of $L_2(D)$ to some $\psi\in H^1(D)$.

Since $\|\psi_n\|_{2,\Omega}\leqq$const., there exists a subsequence of ψ_n converging to some ψ_Ω in the sense of $H^1(\Omega)$, and hence, of $L_2(\Omega)$. But another use of (2.2) shows that the convergence is in $H^2(\Omega'')$ where $\Omega''\subset\subset\Omega$. Therefore, by means of the diagonal process applied to a countable open covering of R^3-K, one can select a subsequence $\psi_{n'}$ which converges in $H^2_{\mathrm{loc}}(R^3-K)$ to a function ψ.

What remains is to prove iii). Since $w_n=\psi_n-c_ne^{ik\cdot x}$ satisfies the radiation condition (1.3), it follows from Green's formula that

$$w_n(x)=\int_{\partial G}\left[\Phi(x,y)\frac{\partial w_n}{\partial\nu_y}-\left(\frac{\partial}{\partial\nu_y}\Phi(x,y)\right)w_n(y)\right]dS_y \tag{2.4}$$
$$-\int_G\Phi(x,y)q(y)w_n(y)dy,\qquad x\in G$$

where $\Phi(x,y)=(4\pi)^{-1}e^{i|k||x-y|}/|x-y|$, ∂G a closed surface of class C^2 lying in $D-K$, G its exterior domain and ν_y the outer normal to ∂G at y. Choose a subsequence n' so that $\psi_{n'}\to\psi$ in $H^2_{\mathrm{loc}}(R^3-K)$ and $c_{n'}\to c\geqq 0$. Then $w_{n'}\to w=\psi-ce^{ik\cdot x}$ in $H^2_{\mathrm{loc}}(R^3-K)$, and hence the traces $w_{n'}|_{\partial G}$ and $(\partial w_{n'}/\partial\nu)|_{\partial G}$ converge in $L_2(\partial G)$ to those of w. Therefore, the first integral of (2.4) converges uniformly in R^3-D.

Let $\varepsilon>0$ be given. Because of the rapid decrease of $q(y)$ and uniform boundedness of w_n in R^3-D, we can find an $R=R_\varepsilon$ so large that

$$\left|\int_{\{|y|>R\}}\Phi(x,y)q(y)[w_n(y)-w(y)]dy\right|<\varepsilon$$

holds for any n and $x\in G$. On the other hand, the integral over $G_R\equiv G\cap\{|y|<R\}$ tends to zero as $n'\to\infty$, because $w_{n'}\to w$ in $L_2(G_R)$ and q is bounded. This shows that $\psi_{n'}$ converges uniformly in R^3-D.

Lemma 2. i) *ψ satisfies the equation* (1.1) *in* R^3-K; ii) $\zeta\psi\in H^2(R^3-K)\cap H^1_0(R^3-K)$ *where ζ is a function described in* (I); iii) *w satisfies the radiation condition.*

Proof. The assertion i) follows from the fact that $\psi_{n'}\to\psi$ in $H^2_{\mathrm{loc}}(R^3-K)$. It follows from (2.3) that $\|\chi_K\psi_n\|_{0,D}\leqq$const. $\sqrt{M/n}\to 0$ as $n\to\infty$. Therefore $\psi=0$ in K, which shows $\zeta\psi\in H^1_0(R^3-K)$. That $\zeta\psi\in H^2(R^3-K)$ is a consequence of the boundary estimate for Δ (see e.g. [4]). Since (2.4) is valid for w, and since $q(x)=O(|x|^{-3-\delta})$, $|x|\to\infty$, it is well known that w satisfies the radiation condition.

Lemma 3.[2)] *The solution of* (I) *is unique. In other words, if* ψ

2) At least in our case, we have only to note that Green's formula gives $\int_{\{|x|<r\}-K}[|\nabla\psi|^2+q|\psi|^2-|k|^2|\psi|^2]dx=\int_{|x|=r}\psi[\partial\bar\psi/\partial|x|]dS=i|k|\int_{|x|=r}|\psi|^2dS+o(1)$ for large r.

satisfies the radiation condition as well as i) *and* ii) *of Lemma* 2, *then* $\psi=0$ *in* R^3-K.

Lemma 4. *There exists a positive constant h such that* $c_n \geqq h$ *for* $n=1, 2, \cdots$.

Proof. Suppose there were a subsequence $c_{n'} \to 0$. By Lemma 1 applied to $\psi_{n'}$[3], there would be a subsequence $\psi_{n''}$ of $\psi_{n'}$ converging to $\psi=w$. Hence, Lemmas 2 and 3 show that $\psi=0$ in the whole space. But this is impossible because of (2.1).

Proof of the theorem. If we choose a subsequence n' of n in such a way that $c_{n'} \to c$ and $\psi_{n'} \to \psi$, then, in virtue of Lemma 4, $c_{n'}^{-1} \to c^{-1} < \infty$ and hence $\varphi_{n'} = c_{n'}^{-1}\psi_{n'} \to c^{-1}\psi = \varphi$. By Lemma 3, the limit φ is unique. This means that every subsequence of φ_n does have a subsequence converging to φ. Therefore, φ_n itself must converge to φ.

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Therefore we have $\int_{|x|=r} |\psi|^2 dS = o(1)$. Then Kato's theorem and the unique continuation theorem on the elliptic equation give $\psi=0$ in R^3-K.

3) All discussions can be carried out even if n of (1.2) is replaced by any a_n diverging to infinity. Thus, Lemma 1 can be applied to any subsequence $\psi_{n'}$ instead of ψ_n.