143. *Bordism Algebra of Involutions*

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1. Introduction. Let $\mathfrak{N}_*$ denote the unoriented Thom bordism ring and let $\mathfrak{N}_*(Z_2)$ denote the unoriented bordism group of fixed point free involutions. Then $\mathfrak{N}_*(Z_2)$ is a free $\mathfrak{N}_*$-module with basis $\{[S^n, a]\}_{n\geq 0}$, where $[S^n, a]$ is the bordism class of the antipodal involution on the n-sphere ([2], Theorem 23.2).

If we regard $\mathfrak{N}_*(Z_2)$ as the bordism group of principal Z_2-bundles over closed manifolds, the tensor product of principal Z_2-bundles induces a multiplication in $\mathfrak{N}_*(Z_2)$, making it an algebra over $\mathfrak{N}_*$. Explicitly, we consider involutions T_1 and T_2 on M_1^m and M_2^n respectively, then both $T_1\times 1$ and $1\times T_2$ induce the same involution T on $M_1^m\times M_2^n/T_1\times T_2$. We have then the multiplication

$$[M_1^m, T_1][M_2^n, T_2]=[M_1^m\times M_2^n/T_1\times T_2, T].$$

J. C. Su [6] stated that $\mathfrak{N}_*(Z_2)$ is an exterior algebra over $\mathfrak{N}_*$ with generators in each dimension 2^n $(n=0,1,2,\cdots)$ and C. S. Hoo [4] showed a multiplicative relation in $\mathfrak{N}_*(Z_2)$ which is equivalent to (2.6) below. In this note, we show the following relation.

Theorem. $[S^{2n+1}, a]=[S^1, a]\cdot\left(\sum_{k=0}^{n}[P^{2k}][S^{2n-2k}, a]\right)$ *for all* n.

As an application we show z_{2k} $(k=1,2,3,\cdots)$ in the following result due to Boardman ([1], Theorem 8.1) is nothing else than $[P^{2k}]=[S^{2k}/a]$:

There exist elements $z_2, z_4, z_5, z_6, z_8, \cdots$ *in* $\mathfrak{N}_*$, *uniquely defined by the condition that*

$$P=w_1+z_2w_1^3+z_4w_1^5+z_5w_1^6+z_6w_1^7+z_8w_1^9+\cdots$$

(*omitting terms of the form* $z_{k-1}w_1^k$ *when* k *is a power of* 2) *is a primitive element in the Hopf algebra* $\mathfrak{N}^*(BO(1))$. *Moreover, these elements* z_k *are a set of polynomial generators for* $\mathfrak{N}_*$.

2. Bordism algebra of involutions. Let us summarize here what is known about $\mathfrak{N}_*$-module $\mathfrak{N}_*(Z_2)$. It has been shown that $\mathfrak{N}_*(Z_2)$ is a free $\mathfrak{N}_*$-module with basis $[S^n, a]$ $(n=0,1,2,\cdots)$, where S^n is an n-sphere and a the antipodal involution on S^n. Let

$$\Delta: \mathfrak{N}_*(Z_2)\to\mathfrak{N}_*(Z_2)$$

be the Smith homomorphism ([2], Theorem 26.1). This is an $\mathfrak{N}_*$-module homomorphism of degree -1, and it can be described as follows. Suppose (M^n, T) is a differentiable fixed point free involution on a closed manifold M^n and $g:(M^n,T)\to(S^N,a)$ is a differentiable equivariant map which is transverse regular on S^{N-1}. Then

$$\Delta([M^n,T])=[g^{-1}(S^{N-1}),T\,|\,g^{-1}(S^{N-1})].$$

From this it is clear that $\Delta([S^n,a])=[S^{n-1},a]$ for all n (with understanding that $[S^k,a]=0$ for $k<0$).

Let $\varepsilon:\mathfrak{N}_*(Z_2)\to\mathfrak{N}_*$ and $\iota:\mathfrak{N}_*\to\mathfrak{N}_*(Z_2)$ be the $\mathfrak{N}_*$-algebra homomorphisms defined by

$$\varepsilon([M^n,T])=[M^n/T]\quad\text{and}\quad\iota([M^n])=[M^n][S^0,a]$$

respectively. Since $\varepsilon\cdot\iota=$identity and $[S^0,a]$ is the identity element of $\mathfrak{N}_*(Z_2)$, so we can identify $\mathfrak{N}_*$ with the subalgebra $\iota(\mathfrak{N}_*)$ of $\mathfrak{N}_*(Z_2)$.

Regard S^{2n+1} as the unit sphere in the complex $(n+1)$-space C^{n+1}, and let

$$\mu:S^1\times S^{2n+1}\to S^{2n+1}$$

be the map defined by $\mu(z,(z_0,z_1,\cdots,z_n))=(zz_0,zz_1,\cdots,zz_n)$.

Proposition 2.1. (C. S. Hoo) *$[S^1,a][S^{2n+1},a]=0$ for all n.*

Proof. C. S. Hoo [4] showed this relation by making use of the involution numbers ([2], 23.1). Here we give a geometrical proof. Let

$$f:S^1\times S^{2n+1}\to S^1\times S^{2n+1}$$

be the map defined by $f(x,y)=(x,\mu(x,y))$. Then

$$f\circ(a\times a)=(a\times 1)\circ f\quad\text{and}\quad f\circ(1\times a)=(1\times a)\circ f.$$

Therefore f induces an equivariant diffeomorphism between

$$(S^1\times S^{2n+1}/a\times a,1\times a)\quad\text{and}\quad((S^1/a)\times S^{2n+1},1\times a),$$

and hence $$[S^1,a][S^{2n+1},a]=[P^1][S^{2n+1},a]=0.$$ q.e.d.

Remark. There is a canonical principal S^1-bundle $P^{2n+1}\to CP^n$. Thus $[P^{2n+1}]=0$ for all n, since P^{2n+1} is the boundary of the associated disk bundle of $P^{2n+1}\to CP^n$.

Lemma 2.2.

(a) $\Delta^2([S^1,a][S^k,a])=[S^1,a][S^{k-2},a]$ *for all* $k\geq 2$,

(b) $\varepsilon\Delta([S^1,a][S^k,a])=0$ *for all* $k\geq 1$.

Proof. Let S_0^{2n}, S^{2n} and S^{2n-1} denote the submanifolds of S^{2n+1} defined by

$$S_0^{2n}=\{(z_0,z_1,\cdots,z_n)\in S^{2n+1}\,|\,z_0\text{ is real}\},$$
$$S^{2n}=\{(z_0,z_1,\cdots,z_n)\in S^{2n+1}\,|\,z_n\text{ is real}\},$$
$$S^{2n-1}=\{(z_0,z_1,\cdots,z_n)\in S^{2n+1}\,|\,z_n=0\}.$$

Then both $\mu:S^1\times S^{2n+1}\to S^{2n+1}$ and $\mu\,|\,S^1\times S_0^{2n}$ are transverse regular on $S^{2n-\varepsilon}$ $(\varepsilon=0,1)$. On the other hand, $\mu\circ(a\times a)=\mu$ and $\mu\circ(1\times a)=a\circ\mu$. Thus μ induces an equivariant map

$$\hat{\mu}:(S^1\times S^{2n+1}/a\times a,1\times a)\to(S^{2n+1},a)$$

which is transverse regular on $S^{2n-\varepsilon}$ $(\varepsilon=0,1)$, and

$$\hat{\mu}^{-1}(S^{2n-1})=(S^1\times S^{2n-1}/a\times a).$$

This shows

$$\Delta^2([S^1, a][S^{2n+1}, a]) = [S^1, a][S^{2n-1}, a]$$

and

$$\Delta^2([S^1, a][S_0^{2n}, a]) = [S^1, a][S_0^{2n-2}, a]$$

for $n \geq 1$. This proves (a).

Next, in general, $\varepsilon\Delta([S^m, a][S^n, a])$ is the bordism class of the hypersurface $H_{m,n}$ which is the subset in $P^m \times P^n$ defined by the equation

$$x_0y_0 + x_1y_1 + x_2y_2 + \cdots + x_py_p = 0,$$

where $p = \min(m, n)$, and $(x_0, x_1, \cdots, x_m)$ and $(y_0, y_1, \cdots, y_n)$ are the standard homogeneous coordinates in P^m and P^n respectively. Then it has been verified by Conner and Floyd ([3], Lemma 2.2) that $[H_{1,k}] = 0$ for all $k \geq 1$. Thus

$$\varepsilon\Delta([S^1, a][S^k, a]) = 0 \quad \text{for } k \geq 1. \qquad \text{q.e.d.}$$

Corollary 2.3. $\Delta^2([S^1, a]x) = [S^1, a]\Delta^2(x)$ *for* $x \in \mathfrak{N}_*(Z_2)$.

Remark. We have relations

(1) $\Delta^4([S^k, a]x) = [S^k, a]\Delta^4(x)$ for $k \leq 3$,

(2) $\Delta^8([S^k, a]x) = [S^k, a]\Delta^8(x)$ *for* $k \leq 7$,

by making use of quaternions and Cayley numbers instead of complex numbers.

Theorem 2.4. $[S^{2n+1}, a] = [S^1, a] \cdot \left(\sum_{k=0}^{n} [P^{2k}][S^{2n-2k}, a] \right)$ *for all* n.

Proof. Put $y_n = [S^{2n+1}, a] + [S^1, a] \cdot \left(\sum_{k=0}^{n} [P^{2k}][S^{2n-2k}, a] \right)$, then $\Delta^2(y_n) = y_{n-1}$ from (2.3). We show $y_n = 0$ by induction on n. It is clear that $y_0 = 0$, so we suppose $y_{n-1} = 0$ for some $n \geq 1$. Since $\Delta^2(y_n) = 0$, we can find x_0, x_1 in $\mathfrak{N}_*$ such that

$$y_n = x_1[S^1, a] + x_0.$$

Then

$$x_0 = \varepsilon(y_n) + x_1\varepsilon([S^1, a]) = 0,$$

and hence

$$y_n = x_1[S^1, a].$$

Next

$$\begin{aligned} x_1 &= \varepsilon\Delta(y_n) \\ &= \varepsilon([S^{2n}, a]) + \sum_{k=0}^{n} [P^{2k}] \cdot \varepsilon\Delta([S^1, a][S^{2n-2k}, a]) \\ &= \varepsilon([S^{2n}, a]) + [P^{2n}] \cdot \varepsilon\Delta([S^1, a][S^0, a]) \\ &= [P^{2n}] + [P^{2n}] = 0 \end{aligned}$$

by (2.2(b)). Thus $y_n = 0$. q.e.d.

Corollary 2.5. $[S^1, a][S^{2n}, a] = \sum_{i=0}^{n} a_{2i}[S^{2n-2i+1}, a]$, *where the element* a_{2i} *in* $\mathfrak{N}_{2i}$ *is defined by* $a_0 = 1$ *and* $\sum_{i=0}^{k} a_{2i}[P^{2k-2i}] = 0$ *for* $k \geq 1$.

Proof.

$$\sum_{i=0}^{n} a_{2i}[S^{2n-2i+1}, a]$$
$$= \sum_{i=0}^{n} a_{2i}[S^1, a]\cdot\left(\sum_{j=0}^{n-i}[P^{2j}][S^{2n-2i-2j}, a]\right)$$
$$= \sum_{k=0}^{n}[S^1, a][S^{2n-2k}, a]\cdot\left(\sum_{i=0}^{k} a_{2i}[P^{2k-2i}]\right)$$
$$=[S^1, a][S^{2n}, a]. \qquad \text{q.e.d.}$$

Corollary 2.6 (C. S. Hoo). $[S^{2m+1}, a][S^{2n+1}, a]=0$.

Proof. This follows from (2.1) and (2.4). q.e.d.

Corollary 2.7. *$\varepsilon\Delta^2([S^{2m+1}, a][S^n, a])=0$ for all m and n.*

Proof. By (2.4), $[S^{2m+1}, a]=[S^1, a]x$ for some x in $\mathfrak{N}_*(Z_2)$. Then, from (2.3),

$$\varepsilon\Delta^2([S^{2m+1}, a][S^n, a])=\varepsilon\Delta^2([S^1, a](x[S^n, a]))$$
$$=\varepsilon([S^1, a]\Delta^2(x[S^n, a]))=[P^1]\varepsilon\Delta^2(x[S^n, a])=0. \qquad \text{q.e.d.}$$

3. Primitive element. In his note, J. M. Boardman stated the following result ([1], Theorem 8.1):

There exist elements $z_2, z_4, z_5, z_6, z_8, \cdots$ in $\mathfrak{N}_*$, uniquely defined by the condition that

$$P=w_1+z_2w_1^3+z_4w_1^5+z_5w_1^6+z_6w_1^7+z_8w_1^9+\cdots$$

(omitting terms of the form $z_{k-1}w_1^k$ when k is a power of 2) is a primitive element in the Hopf algebra $\mathfrak{N}^*(BO(1))$. Moreover, these elements z_k are a set of polynomial generators for $\mathfrak{N}_*$.

Put $z_0=[P^0]$, then we can state the following result which is essentially proved by M. Kamata [5], so we omit the proof.

Lemma 3.1. *Let $\alpha_i(m, n)$ be an element in $\mathfrak{N}_{m+n-i}$, defined by the condition*

$$[S^m, a][S^n, a]=\sum_{i=0}^{m+n}\alpha_i(m, n)[S^i, a].$$

Then the following relations hold.

(a) $\sum_{i\geq 1} z_{i-1}\alpha_{k+i}(m, n)=\sum_{i\geq 1} z_{i-1}(\alpha_k(m-i, n)+\alpha_k(m, n-i))$ *for all* $k\geq 0$,

(b) (J. C. Su) $\alpha_{m+n}(m, n)=\binom{m+n}{m}$ mod 2,

(c) $\alpha_0(m, n)=[P^m][P^n]+\sum_{i=1}^{m+n}\alpha_i(m, n)[P^i]$.

Theorem 3.2. *$z_{2k}=[P^{2k}]$ for all k.*

Proof. $\alpha_k(1, 2n+1)=0$ for all n and k, since

$$[S^1, a][S^{2n+1}, a]=0$$

by (2.1). Hence

(3.2.1) $$\alpha_1(0, 2n+1)+\sum_{i\geq 0} z_{2i}\alpha_1(1, 2n-2i)=0,$$

from (3.1(a)). On the other hand,

$$[S^{2n+1}, a]=\sum_{i=0}^{n}[S^1, a][S^{2n-2i}, a][P^{2i}]$$
$$=\sum_{i=0}^{n}\sum_{k=0}^{2n-2i+1}\alpha_k(1, 2n-2i)[P^{2i}][S^k, a],$$

from (2.4). Since $\mathfrak{N}_*(Z_2)$ is a free $\mathfrak{N}_*$-module with basis $\{[S^k, a]\}$, we have

$$\sum_{i=0}^{n}\alpha_1(1, 2n-2i)[P^{2i}]=0 \qquad \text{for } n\geq 1. \tag{3.2.2}$$

Then we have a desired result, by induction, from (3.2.1), (3.2.2), $\alpha_1(0,1)=1$ and $\alpha_1(0, 2n+1)=0$ for $n\geq 1$. q.e.d.

Remark. We could not determine the elements z_{2k+1}, but if we use the relation (3.1), we are able to calculate z_{2k+1}. For example,

$$z_5=[H_{2,4}],$$
$$z_9=[H_{2,8}]+[H_{2,4}][P^2]^2,$$
$$z_{11}=[H_{4,8}]+[H_{2,4}][P^6]+[H_{2,4}][P^4][P^2].$$

The elements $\alpha_i(m, n)$ are also calculable from (3.1). For example,

$$[S^2, a][S^4, a]=[S^6, a]+[H_{2,4}][S^1, a]+[P^6]+[P^4][P^2],$$
$$[S^3, a][S^6, a]=[P^2][S^7, a]+[P^6][S^3, a].$$

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