215. On an Algebraic Model for von Neumann Algebras

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1. Recently N. Dinculeanu and C. Foiaş [2] introduced the concept of algebraic models for probability measures in their researches on conjugacy of measure preserving transformations.

Since the theory of von Neumann algebras is recognized as a non-commutative extension of the measure theory, we can expect that the theory of Dinculeanu and Foiaş has an analogue for von Neumann algebras. In the present note, we shall engage in this direction.

2. Let (Γ, φ) be a pair of a group Γ and a complex function φ of positive type defined on Γ. Then we shall call (Γ, φ) a *measure system* provided that $\varphi(\gamma)=1$ if and only if $\gamma=1$. Especially, in case that Γ is abelian, our notion coincides with that of Dinculeanu-Foiaş. Two measure systems (Γ, φ) and (Γ', φ') are said to be *isomorphic* if there exists an isomorphism ϕ of Γ onto Γ' such that $\varphi(\gamma)=\varphi'(\phi\gamma)$ for γ in Γ.

Now we shall introduce the notion of an algebraic model for a von Neumann algebra which is a modification of that of Dinculeanu-Foiaş:

Definition 1. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ with a generating vector x. A measure system (Γ, φ) is an *algebraic model* for $\mathcal{A}$, if there exists an isomorphism J of Γ into the unitary group of $\mathcal{A}$ such that

(i) $J\Gamma$ generates $\mathcal{A}$,

and

(ii) $\varphi(\gamma)=(J\gamma x \mid x)$, for γ in Γ.

It is clear that the unitary group $\Gamma(\mathcal{A})$ itself is an algebraic model for $\mathcal{A}$ if x is separating.

Let us suppose that (Γ, φ) is a measure system. Since φ is positive definite, the theorem of Gelfand and Raikov (cf. [3; p. 393]) gives a unitary representation π of Γ on a Hilbert space $\mathfrak{H}_\varphi$ induced by φ such that

$$\varphi(\gamma)=(\pi(\gamma)\xi \mid \xi) \tag{1}$$

for every γ in Γ, where ξ is a generating vector for $\pi(\Gamma)$. Since $\varphi(\gamma)=1$ if and only if $\gamma=1$, π is an injective map. In fact, if $\pi(\gamma_1)=\pi(\gamma_2)$ for $\gamma_1, \gamma_2 \in \Gamma$, then $\pi(\gamma_1\gamma_2^{-1})=\mathrm{I}$, so that $\varphi(\gamma_1\gamma_2^{-1})=1$ by (1) or $\gamma_1=\gamma_2$. Let $\mathcal{A}(\Gamma, \varphi)$ denote the von Neumann algebra generated by $\{\pi(\gamma) \mid \gamma \in \Gamma\}$. Then we have the following theorem:

Theorem 1. *Let $\mathcal{A}$ be a von Neumann algebra acting on $\mathfrak{H}$ with* *a cyclic vector. If (Γ, φ) is an algebraic model for $\mathcal{A}$, then $\mathcal{A}$ is spatially isomorphic to $\mathcal{A}(\Gamma, \varphi)$.*

Proof. Since (Γ, φ) is an algebraic model for $\mathcal{A}$, there exist an isomorphism J of Γ into $\Gamma(\mathcal{A})$ and a separating and generating unit vector x in $\mathfrak{H}$ satisfying (i) and (ii) of Definition 1. Put

$$V[\pi(\gamma)\xi] = J(\gamma)x, \qquad (\gamma \in \Gamma). \tag{2}$$

Then it is easy to observe that V is an isometry of $\{\pi(\gamma)\xi \mid \gamma \in \Gamma\}$ onto $\{J(\gamma)x \mid \gamma \in \Gamma\}$. Next we define

$$V[\alpha\pi(\gamma_1)\xi + \beta\pi(\gamma_2)\xi] = \alpha J(\gamma_1)x + \beta J(\gamma_2)x,$$

for $\gamma_1, \gamma_2 \in \Gamma$ and $\alpha, \beta \in \boldsymbol{C}$, where $\boldsymbol{C}$ is the complex number field. Then V is well-defined on the linear span of $\{\pi(\gamma)\xi \mid \gamma \in \Gamma\}$. Since

$$\begin{aligned}
&\|\alpha J(\gamma_1)x + \beta J(\gamma_2)x\|^2 \\
&= |\alpha|^2(J(1)x \mid x) + \alpha\bar{\beta}(J(\gamma_2^{-1}\gamma_1)x \mid x) + \bar{\alpha}\beta(J(\gamma_1^{-1}\gamma_2)x \mid x) + |\beta|^2(J(1)x \mid x \\
&= |\alpha|^2\varphi(1) + \bar{\alpha}\beta\varphi(\gamma_2^{-1}\gamma_1) + \alpha\bar{\beta}\varphi(\gamma_1^{-1}\gamma_2) + |\beta|^2\varphi(1) \\
&= |\alpha|^2(\pi(1)\xi \mid \xi) + \bar{\alpha}\beta(\pi(\gamma_2^{-1}\gamma_1)\xi \mid \xi) + \alpha\bar{\beta}(\pi(\gamma_1^{-1}\gamma_2)\xi \mid \xi) + |\beta|^2(\pi(1)\xi \mid \xi) \\
&= \|\alpha\pi(\gamma_1)\xi + \beta\pi(\gamma_2)\xi\|^2,
\end{aligned}$$

V is an isometry from the linear span of $\{\pi(\gamma)\xi \mid \gamma \in \Gamma\}$ onto the linear span of $\{J(\gamma)x \mid \gamma \in \Gamma\}$. Since ξ and x are cyclic vectors for $\pi(\Gamma)$ and $\mathcal{A}$ respectively, and since $J(\Gamma)$ (which contains the unit element of $\mathcal{A}$) generates $\mathcal{A}$, V can be uniquely extended to an isometry of $\mathfrak{H}_\varphi$ onto $\mathfrak{H}$, which is also denoted by V.

For any fixed $\gamma \in \Gamma$ and every $\delta \in \Gamma$, we have

$$V\pi(\gamma)V^{-1}J(\delta)x = V\pi(\gamma)\pi(\delta)\xi = V\pi(\gamma\delta)\xi = J(\gamma\delta)x = J(\gamma)J(\delta)x.$$

Since x is cyclic for $\mathcal{A}$ and $J(\Gamma)$ generates $\mathcal{A}$, we have

$$V\pi(\gamma)V^{-1} = J(\gamma), \tag{3}$$

for any $\gamma \in \Gamma$. Hence we have by (3),

$$V\sum_{i=1}^{n}\alpha_i\pi(\gamma_i)V^{-1} = \sum_{i=1}^{n}\alpha_i J(\gamma_i), \tag{4}$$

for $\gamma_i \in \Gamma$ and $\alpha_i \in \boldsymbol{C}\,(1 \leqq i \leqq n)$.

Since $A \in \mathcal{A}(\Gamma, \varphi)$ is the weak limit of some net $\{A_\alpha\}$, where

$$A_\alpha = \sum_{i=1}^{m_\alpha}\beta_i^{(\alpha)}\pi(\gamma_i^{(\alpha)}),$$

for some $\gamma_i^{(\alpha)} \in \Gamma$ and $\beta_i^{(\alpha)} \in C\,(1 \leqq i \leqq n_\alpha)$, and $VA_\alpha V^{-1}$ converges weakly to VAV^{-1},

$$\Phi(A) = VAV^{-1}$$

gives a spatial isomorphism of $\mathcal{A}(\Gamma, \varphi)$ onto $\mathcal{A}$.

By Theorem 1, we can obtain a non-abelian extension of the theorem of Dinculeanu and Foiaş:

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras acting on $\mathfrak{H}$ and $\mathfrak{K}$ with a generating vector respectively. Suppose that (Γ, φ) and (Γ', φ') be algebraic models for $\mathcal{A}$ and $\mathcal{B}$ respectively. If (Γ, φ) and (Γ', φ') are isomorphic, then $\mathcal{A}$ and $\mathcal{B}$ are spatially isomorphic.*

Proof. By Theorem 1, we have that $\mathcal{A}$ and $\mathcal{B}$ are spatially isomorphic with $\mathcal{A}(\Gamma, \varphi)$ and $\mathcal{A}(\Gamma', \varphi')$ respectively. By the hypothesis, there exists an isomorphism ϕ of Γ and Γ' such that

$$\varphi(\gamma)=\varphi'(\phi\gamma), \qquad \text{for } \gamma \in \Gamma.$$

Therefore,

$$(\pi(\gamma)\xi \mid \xi)=(\pi'(\phi\gamma)\xi' \mid \xi'), \qquad \gamma \in \Gamma, \tag{5}$$

where π and π' are injective unitary representations of Γ and Γ' respectively and ξ and ξ' are cyclic unit vectors for π and π' respectively. Using (5), as same as the proof of Theorem 1, we can prove that $\mathcal{A}(\Gamma, \varphi)$ and $\mathcal{A}(\Gamma', \varphi')$ are spatially isomorphic. Hence by Theorem 1 $\mathcal{A}$ and $\mathcal{B}$ are spatially isomorphic.

3. Now we shall discuss the abelian case. Let (X, Σ, μ) be a probability space. We shall denote by $\Gamma(\mu)$ the set of all functions f in $L^\infty(\mu)$ with $|f|=1$. The following definition is due to Dinculeanu and Foiaş:

Definition 2. A measure system (Γ, φ) is said to be an *algebraic model* for a measure μ if there exists an isomorphism J of Γ into $\Gamma(\mu)$ such that

(a) $J(\Gamma)$ spans $L^2(\mu)$,

and

(b) $\varphi(\gamma)=\int J\gamma d\mu, \qquad$ for each $\gamma \in \Gamma$.

It is well known that $L^\infty(\mu)$ is a maximal abelian von Neumann algebra acting on $L^2(\mu)$, and the identity function 1 in $L^2(\mu)$ is a generating and separating vector for $L^\infty(\mu)$. It is evident $\Gamma(\mu)=\Gamma(L^\infty(\mu))$. If (Γ, φ) is an algebraic model for μ, there exists an isomorphism J of Γ into $\Gamma(\mu)$ satisfying (a) and (b) of Definition 2. Since (a) implies that $J(\Gamma)$ generates $L^\infty(\mu)$, and (b) implies

$$\varphi(\gamma)=(J(\gamma)1 \mid 1), \qquad \text{for } \gamma \in \Gamma,$$

it is shown that (Γ, φ) is an algebraic model for $L^\infty(\mu)$ in the sense of Definition 1. By Theorem 2, we have the following theorem:

Theorem 3 (Dinculeanu-Foiaş). *Let (Γ, φ) and (Γ', φ') be algebraic models for measures μ and μ' respectively. If (Γ, φ) and (Γ', φ') are isomorphic, then $L^\infty(\mu)$ and $L^\infty(\mu')$ are spatially isomorphic.*

References

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[2] N. Dinculeanu and C. Foiaş: Algebraic models for measures. Illinois J. Math., **12**, 340–351 (1968).

[3] M. A. Naimark: Normed Rings. Noordhoff, Groningen (1959).