214. On Multi-Valued Mappings and Generalized Metric Spaces

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Quite a few different kinds of spaces like 1-st countable spaces, sequential spaces and M-spaces have been characterized as images or inverse images of metric spaces by unique-valued mappings satisfying certain conditions, but so far little efforts have been made to take advantage of multi-valued mappings in this aspect of study. The purpose of the present paper is to show usefulness of such mappings in characterization of two interesting classes of generalized metric spaces, σ-spaces (due to [4]) and M^*-spaces (due to [2]). Throughout the paper spaces are at least T_1. As for general terminologies and symbols the reader is refered to [3]. Some results, terminologies and references concerning multivalued mappings will be found in [1]. We also use Theorem 1 of [5] in the following discussions.

Definition 1. Let f be a multi-valued mapping from a space X to a space Y such that $f(x)\neq\emptyset$ for every $x\in X$, and $f^{-1}(y)\neq\emptyset$ for every $y\in Y$. (Such a mapping will be called simply a *map* from on.) Then for each subset C of X and for each subset D of Y we define the following symbols.

$$f(C)=\cup\{f(x)\mid x\in C\},\ f\langle C\rangle=\{y\mid y\in Y, f^{-1}(y)\subset C\},$$
$$f^{-1}(D)=\cup\{f^{-1}(y)\mid y\in D\},\ f^{-1}\langle D\rangle=\{x\mid x\in X, f(x)\subset D\}.$$

Then f is called *closed* if $f(C)$ is closed in Y for every closed in Y for every closed subset C of X. If for each $y\in Y$ there is $x\in f^{-1}(y)$ such that $f^{-1}\langle V\rangle$ is a nbd(=neighborhood) of x for every nbd V of y, then the map f is called *selection continuous.* If for each $y\in Y$ and for every nbd V of y there is $x\in f^{-1}(y)$ such that $f^{-1}\langle V\rangle$ is a nbd of x, then f is called *w. selection continuous.* It is obvious that for one-valued mappings our definition of closed map turns out to be the ordinary one, and 'selection continuous' as well as 'w. selection continuous' coincide with 'continuous' in the ordinary sense. Furthermore f will be called a *s. perfect map* if it is closed, selection continuous and compact, i.e. $f^{-1}(y)$ is compact for every $y\in Y$.

Proposition 1. *Let X be a regular σ-space and Y a space. If there is a closed, w. selection continuous map f from X to Y, then Y is σ.*

Proof. Let $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}_n$ be a σ-closure preserving net (=network) for X consisting of closed sets. Then $f(\mathcal{U})=\{f(U)\mid U\in\mathcal{U}\}$ is obviously a σ-closure preserving collection of closed sets in Y. We claim that it is a net for Y. Let y be a point of Y and V an open nbd of y. Then there is $x\in f^{-1}(y)$ such that $f^{-1}\langle V\rangle$ is a nbd of x. Hence there is $U\in\mathcal{U}$ such that $x\in U\subset f^{-1}\langle V\rangle$. Then $f(U)\in f(\mathcal{U})$, and $y\in f(U)\subset V$. Thus $f(\mathcal{U})$ is a net, and hence Y is σ.

Theorem 1. *A regular space Y is a σ-space if and only if one of the following conditions is satisfied.*

i) *There is a metric space X and a closed, w. selection continuous map from X to Y.*

ii) *There is a metric space X and a closed, selection continuous map from X to Y.*

iii) *There is a subspace X of Baire's zero-dimensional space $N(A)$ and a s. perfect map from Y to Y.*

Proof. It suffices, by virtue of Proposition 1, to show the necessity of the condition iii). Let Y be a σ-space; then there is a σ-discrete net $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}_n$, where each $\mathcal{U}_n$ is a discrete collection of closed sets. Let $\mathcal{W}_n=\mathcal{U}_n\cup\{X\}, n=1,2,\cdots$. Then $\mathcal{W}_n$ is a locally finite closed cover of order 2. Put $\mathcal{W}_n=\{W_\alpha\mid\alpha\in A_n\}$, and $A=\bigcup_{n=1}^{\infty}A_n$. By defining that $W_\alpha^n=\emptyset$ for every $\alpha\in A-A_n$, we may assume $\mathcal{W}_n=\{W_\alpha^n\mid\alpha\in A\}$. Now, define a subspace X of the Baire's zero-dimensional space $N(A)$ by

$$X=\left\{(\alpha_1,\alpha_2,\cdots)\in N(A)\,\middle|\,\bigcup_{i=1}^{\infty}W_{\alpha_i}^i\neq\emptyset\right\}W.$$

Then we define a map f from X to Y by

$$f((\alpha_1,\alpha_2,\cdots))=\bigcap_{i=1}^{\infty}W_{\alpha_i}^i,(\alpha_1,\alpha_2,,,,)\in X.$$

Obviously $f(x)\neq\emptyset$ for each $x\in X$, and $f^{-1}(y)\neq\emptyset$ for each $y\in Y$. To prove that f is selection continuous, let $y\in Y$; then there is $\alpha_n\in A_n$, $n=1,2,\cdots$ such that $\{W_{\alpha_n}^n\mid n=1,2,\cdots\}$ is a net about y. Put $x=(\alpha_1,\alpha_2,\cdots)$; then $x\in f^{-1}(y)$ because $f(x)=\bigcap_{n=1}^{\infty}W_{\alpha_n}^n\ni y$. Let V be a given nbd of y in Y; then there is n for which $W_{\alpha_n}^n\subset V$. Thus $N(\alpha_1,\cdots,\alpha_n)=\{(\beta_1,\beta_2,\cdots)\in X\mid\beta_1=\alpha_1,\cdots,\beta_n=\alpha_n\}$ is a nbd of x in X such that $f(N(\alpha_1,\cdots,\alpha_n))\subset\bigcap_{i=1}^{n}W_{\alpha_i}^i\subset V$. Therefore $N(\alpha_1,\cdots,\alpha_n)\subset f^{-1}\langle V\rangle$, proving that $f^{-1}\langle V\rangle$ is a nbd of x. Thus f is selection continuous.

Our next claim is that f is closed. Observe that each closed set of X is expressed as $\bigcap_{n=1}^{\infty}F_n$, where each F_n is a union of closed sets of the form $N(\alpha_1,\cdots,\alpha_n)$. Now, let us prove that $f\left(\bigcap_{n=1}^{\infty}F_n\right)$ is a closed set in Y. Suppose $y\notin f\left(\bigcap_{n=1}^{\infty}F_n\right)$; then $y\notin f(F_1\cap\cdots\cap F_n)$ for some n.

Because, otherwise for each n there is $x_n \in F_1 \cap \cdots \cap F_n$ such that $y \in f(x_n)$. Let $x_1=(\alpha_1^1, \cdots), x_2=(\alpha_1^2, \alpha_2^2, \cdots), x_3=(\alpha_1^3, \alpha_2^3, \alpha_3^3, \cdots), \cdots$; then we obtain

$$N(\alpha_1^1) \subset F_1,$$
$$N(\alpha_1^2) \subset F_1, N(\alpha_1^2, \alpha_2^2) \subset F_2,$$
$$N(\alpha_1^3) \subset F_1, N(\alpha_1^3, \alpha_2^3) \subset F_2, N(\alpha_1^3, \alpha_2^3, \alpha_3^3) \subset F_3, \cdots$$

Since $y \in W^1_{\alpha_1^i} \in \mathcal{W}_1, i=1,2,\cdots, \alpha_1^i$ can take on at most two different values. Thus $\alpha_1^{i_{11}}=\alpha_1^{i_{12}}=\cdots=\beta_1$ for some infinite subsequence $\{i_{11}, i_{12}, \cdots\}$ of $\{1,2,\cdots\}$. Similarly we can choose an infinite subsequence $\{i_{21}, i_{22}, \cdots\}$ from $\{i_{11}, i_{12}, \cdots\}$ such that $\alpha_2^{i_{21}}=\alpha_2^{i_{22}}=\cdots=\beta_2$. Repeating the same process, generally we choose a subsequence $\{i_{k1} \cdot i_{k2}, \cdots\}$ of $\{i_{k-1}, i_{k-12}, \cdots\}$ such that $\alpha_k^{i_{k1}}=\alpha_k^{i_{k2}}=\cdots=\beta_k$. Now $\{i_{11}, i_{21}, i_{31}, \cdots\}$ is a subsequence of $\{1,2,\cdots\}$ such that $\alpha_1^{i_{11}}=\beta_1$; $\alpha_1^{i_{21}}=\beta_1, \alpha_2^{i_{21}}=\beta_2$; $\alpha_1^{i_{31}}=\beta_1, \alpha_2^{i_{31}}=\beta_2, \alpha_3^{i_{31}}=\beta_3$; $\cdots$. Then $x=(\beta_1, \beta_2, \cdots)$ is a point of X. Note that $y \in \bigcap_{i=1}^{\infty} W^i_{\beta_i}=f(x)$ and also that $N(\beta_1) \subset F_1, N(\beta_1, \beta_2) \subset F_2, \cdots$, which implies $x \in N(\beta_1) \cap N(\beta_1, \beta_2) \cap \cdots \subset \bigcap_{n=1}^{\infty} F_n$. Therefore $y \in f\left(\bigcap_{n=1}^{\infty} F_n\right)$, which is a contradiction. Thus we can assume $y \notin f(F_1 \cap \cdots \cap F_n)$ for some n. Observe that $F_1 \cap \cdots \cap F_n$ is a sum of closed sets of the form $N(\gamma_1^1) \cap \cdots \cap N(\gamma_1^n, \cdots, \gamma_n^n)$. If $\gamma_1^1=\gamma_1^2=\cdots=\gamma_1^n, \gamma_2^2=\cdots=\gamma_2^n, \cdots, \gamma_{n-1}^{n-1}=\gamma_{n-1}^n$, then $f(N(\gamma_1^1) \cap \cdots \cap N(\gamma_1^n, \cdots, \gamma_n^n))=f(N(\gamma_1^n, \cdots, \gamma_n^n))=W^1_{\gamma_1^n} \cap \cdots \cap W^n_{\gamma_n^n}$. Otherwise $f(N(\gamma_1^1) \cap \cdots \cap N(\gamma_1^n, \cdots, \gamma_n^n))=\emptyset$. Therefore $f(F_1 \cap \cdots \cap F_n)$ is a sum of closed sets of the form $W^1_{\gamma_1} \cap \cdots \cap W^n_{\gamma_n}$. Since $\mathcal{W}_1, \cdots, \mathcal{W}_n$ are locally finite, $f(F_1 \cap \cdots \cap F_n)$ is a closed set. Hence $X-f(F_1 \cap \cdots \cap F_n)$ is a nbd of y disjoint from $f\left(\bigcap_{n=1}^{\infty} F_n\right)$. Thus $f\left(\bigcap_{n=1}^{\infty} F_n\right)$ is a closed set.

To show that f is a compact map, let $y \in Y$, and $x_1, x_2, \cdots \in f^{-1}(y)$. Then by the same argument as before we choose $x=(\beta_1, \beta_2, \cdots)$ in $f^{-1}(y)$. We also claim that x is a cluster point of the sequence $\{x_i \mid i=1, 2, \cdots\}$, because for each n there is $x_{k(n)}$ whose first n coordinates coincide with those of x. Since X is a metric space, this means that $f^{-1}(y)$ is compact.

Definition 2. Let f be a map from X to Y. If $f^{-1}(G)$ is closed in X for every closed set G in Y, then f is called *continuous*. If f is continuous and if $f^{-1}(y)$ is compact (countably compact) for each point y of Y, then f is *perfect* (*quasi-perfect*). If $f(x)$ is countably compact for each point x of X, then f is *Y-countably compact*.

Proposition 2. *Let f be a quasi-perfect, Y-countably compact map from X to Y. If X is an M^*-space, then so is Y.*

Proof. X has a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \cdots$ of locally finite closed covers satisfying the following condition:

(M) If $x_i \in S(x_0, \mathcal{U}_i), i=1,2,\cdots$ for a fixed point x_0, then $\{x_i\}$ has a cluster point.

Then it is easy to see that $f(\mathcal{U}_i), i=1,2,\cdots$ are closure-preserving, point-finite and accordingly locally finite closed covers of Y. Let $G_1 \supset G_2 \supset \cdots$ be a sequence of non-empty closed sets in Y such that $G_i \subset S(y_0, f(\mathcal{U}_i))$. Then, put $H_i = f^{-1}(G_i) \cap S(f^{-1}(y_0), \mathcal{U}_i)$ to get a decreasing sequence $\{H_i \mid i=1,2,\cdots\}$ of non-empty closed sets in X. Assume $\bigcap_{i=1}^{\infty} H_i = \emptyset$ to prove the contrary. Then for each $x \in f^{-1}(y_0)$ there is $i(x)$ such that $S(x, \mathcal{U}_{i(x)}) \cap H_{i(x)} = \emptyset$, because otherwise from the condition (M) it follows that $\bigcap_{i=1}^{\infty} H_i \neq \emptyset$. Let $U_i(x) = X - \cup\{U \in \mathcal{U}_i \mid x \notin U\}$; and $V_i = \cup\{U_{i(x)}(x) \mid x \in f^{-1}(y_0)$ and $i(x)=i\}$. Then each V_i is open, and $\bigcap_{i=1}^{\infty} V_i \supset f^{-1}(y_0)$. Since $f^{-1}(y_0)$ is countably compact, $\bigcup_{i=1}^{k} V_i \supset f^{-1}(y_0)$ for some k. Thus each point x' of $f^{-1}(y_0)$ belongs to V_i for some $i \leqq k$, i.e. $x' \in U_i(x)$ for some x with $i(x)=i$. This implies that each element U of $\mathcal{U}_i$ contains x whenever it contains x'. Therefore $S(x', \mathcal{U}_k) \subset S(x', \mathcal{U}_i) \subset S(x, \mathcal{U}_i) \subset X - H_i \subset X - H_k$. Thus $S(f^{-1}(y_0), \mathcal{U}_k) \cap H_k = H_k = \emptyset$, which is a contradiction. Hence we conclude that $x \in \bigcap_{i=1}^{\infty} H_i = \emptyset$. Now $f(x) \cap G_i \neq \emptyset, i=1,2,\cdots$, and $f(x)$ is countably compact. Hence $\bigcap_{i=1}^{\infty} G_i \neq \emptyset$, proving that Y is an M^*-space.

Theorem 2. *A space Y is an M^*-space if and only if it satisfies one of the following conditions.*

i) *There is a metric space X and a quasi-perfect, Y-countably Y-compact map from X to Y.*

ii) *There is a metric space X and a perfect, Y-countably compact map from X to Y.*

iii) *There is a subspace X of $N(A)$ and a perfect, Y-countably compact map from X to Y.*

Proof. It suffices, by virtue of Proposition 2, to prove the necessity of iii). Let $\mathcal{U}_1 > \mathcal{U}_2 > \cdots$ be a sequence of locally finite closed covers of Y satisfying (M). Let $\mathcal{U}_n = \{U_\alpha \mid \alpha \in A_n\}$, and $A = \bigcup_{i=1}^{\infty} A_n$. Define a subspace X of the Baire's 0-dimensional space $N(A)$ and a map f from X to Y by $X = \{(\alpha_1, \alpha_2, \cdots) \in N(A) \mid \alpha_i \in A_i, \bigcap_{i=1}^{\infty} U_{\alpha_i} \neq \emptyset\}$, $f((\alpha_1, \alpha, _2 \cdots)) = \bigcap_{i=1}^{\infty} U_{\alpha_i}$. Then $f(x)$ is obviously countably compact for each $x \in X$. To prove the continuity of f, suppose G is a closed set in Y, and $x = (\alpha_1, \alpha_2, \cdots) \notin f^{-1}(G)$ in X. Then $f(x) = \bigcap_{i=1}^{\infty} U_{\alpha_i} \subset Y - G$. Therefore by $(M) U_{\alpha_1} \cap \cdots \cap U_{\alpha_i} \subset Y - G$ for some i, and hence $N(\alpha_1, \cdots, \alpha_i)$ is a nbd of x whose image is disjoint from G. Thus $N(\alpha_1, \cdots, \alpha_i) \cap f^{-1}(G) = \emptyset$, proving that $f^{-1}(G)$ is closed. The rest of the proof is similar to the proof of Theorem 1.

Remark. As suggested by Proposition 2 which generalizes two theorems of T. Ishii [2] at the same time, another advantage of multi-valued maps is to provide us with possibility to unify two different types of theories of unique-valued mappings, theory of images and that of inverse images.

References

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