213. A Remark on the Concept of Channels. II

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In the previous note [5], the concept of generalized channels is introduced. In the present note, the effect of the action of a motion on the input will be discussed. Incidentally, the deformation of the spectra of operators through a generalized channel will be considered.

1. Following after the notation of Dixmier [4], the subconjugate space A_* of a von Neumann algebra A is the Banach space of all ultraweakly continuous linear functionals defined on A. A *generalized channel* K is a positive linear transformation defined on a von Neumann algebra B, say *output*, with the range in a von Neumann algebra A, say *input*, which preserves the identity; in other words, the subconjugate K_* of K is positive and norm preserving:

(1) $$\|K_*\rho\| = \|\rho\|,$$

for $\rho \geqq 0$, cf. [5]. Conveniently, K_* will be called a generalized channel too. A generalized channel K_* transfers a normal state ρ from A_* to B_*, and $K_*\rho$ is a normal state of B. If $A=B$, then a generalized channel K will be called a *transition*; if A is abelian then a transition is a transition operator in probability.

2. The concept of generalized channels is born on the information theory, but it is not restricted. Suppose that the input A represents a physical system and the output an observation instrument. A state of the physical system will drive some state of the instrument, if they are connected together. Thus a generalized channel can be considered as a mathematical model for physical measurements. Especially, the situation is suitable for statistical mechanics, including both classical and quantum.

A *motion* μ of a system A is a (*-preserving) automorphism of a von Neumann algebra A, according to a modification of the definition of Segal [8]. A motion μ is ultraweakly continuous; hence the subconjugate (may be abbreviated by μ too) of the motion transforms a normal state ρ to a normal state ρ^μ by

(2) $$\rho^\mu(a) = \rho(a^\mu),$$

for every $a \in A$.

What happens for the receiver if a motion acts on input? The observer obtained $K_*\rho$ before the motion through the channel K. After the motion, he receives $K_*\rho^\mu$. Put

$$(3) \qquad (K_*\rho)^\nu = M_*(K_*\rho) = K_*\rho^\mu.$$

Then ν (or M_*) is itself a generalized channel with the equal input and output B; for any $\rho \geqq 0$, ν satisfies

$$\|(K_*\rho)^\nu\| = \|K_*\rho^\mu\| = \|\rho^\mu\| = \|\rho\| = \|K_*\rho\|,$$

by (1). This shows;

I. *A motion on the input induces a transition on the output.*

In the case of macroscopic measurements, von Neumann [7; V. 4] analysed that the output of macroscopic measurement is abelian and finite dimensional; hence one can define that the output B is *macroscopic* if B is abelian and finite dimensional. In this case, transition can be described by a Markov matrix on the character space (=pure state space) of the algebra B. Hence I implies

II. *A motion on the input induces a Markov chain on the character space of the output if the output is macroscopic.*

Before to proceed further, it may be remarked that II remains true if the input is replaced by a C^*-algebra: In this case, a motion of a C^*-algebra is an automorphism, and a generalized channel for C^*-algebras is a positive linear transformation K preserving the identity which maps the output B into the input A; hence the state space $\sum_A$ of A is mapped by K into the state space $\sum_B$ of B, cf. [8].

II proposes a satisfactory foundation for statistical mechanics: For a macroscopic (=statistical) observer observes a Markov chain on the character space of the instrument driven by a motion of the physical system. If the inverse μ^{-1} of μ induces the same Markov matrix (that is, the reversibility of the motion is assumed), then the Markov matrix is symmetric; hence the entropy for the observer increases time to time. If a suitable ergodicity is assumed for ν, then ν drives any state of B rapidly to the equilibrium state which has the maximum entropy; this is Gibbs' H-Theorem in statistical mechanics. The details are omitted.

If the generalized channel K perturbes the input, then the situation is not so simple, which will be discussed in another occasion as a continuation of [3] and [6].

3. Recently in [1], Berberian gives a determinant-free proof of a conjecture of von Neumann (which is proved originally by Fuglede and Kadison with their determinant theory): In a finite factor with the trace τ, the convex hull co $\sigma(a)$ of the spectrum $\sigma(a)$ of an element a contains $\tau(a)$. It seems that Berberian's proof is an eminent improvement in the theory of finite factors. In his proof, he points out that the closed numerical range $\bar{W}(a^\natural)$ of $a^\natural$ is contained in the convex hull of the spectrum of a, where $\natural$ is Dixmier's center-valued trace; consequently, $\sigma(a^\natural) \subset \text{co}\,\sigma(a)$. Since Dixmier's trace is the conditional expectation conditioned by the center in the sense of Umegaki [11], one naturally ask: In a von Neumann algebra A and the conditional expectation ε conditioned by a von Neumann subalgebra B, is it true that

(4) $$\sigma(a^{\varepsilon})\subset \mathrm{co}\,\sigma(a)$$

for every $a\in A$? In general, the conjecture is false; H. Choda presents an example in a seminar talk:

III (H. Choda). *Even if B is abelian,* (4) *is not true in general.*

It is sufficient to disprove (4) that a finite factor A contains an element a and an abelian subalgebra B which satisfy $a^2=0$ and a^{ε} is a non-zero hermitean element of B: This is the case if A is all 2×2 matrices, B is the diagonal,

$$a=\begin{pmatrix} 1 & 1\\ -1 & -1\end{pmatrix} \quad\text{and}\quad a^{\varepsilon}=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

However, if a satisfies a certain additional condition, then (4) will be proved by the technique created by Berberian [1]. For example, one has

IV. *If a is convexoid* (i.e., $\bar{W}(a)=\mathrm{co}\,\sigma(a)$), *then* (4) *is true for a.*

Since the conditional expectation is a generalized channel, IV is a consequence of the following

V. *If K is a generalized channel, and if b is a convexoid belonging to the output B, then*

(5) $$\sigma(Kb)\subset \mathrm{co}\,\sigma(b).$$

To prove V, one needs Berberian-Orland's theorem [2] (which is implicitly contained in a theorem of Takeda [10; Theorem 1], cf. also [9]): if Σ is the state space of a C^*-algebra A, then

(6) $$\bar{W}(a)=\Sigma(a)=\{\rho(a)\mid \rho\in\Sigma\},$$

for every $a\in A$. If $\rho\in\sum_A$ then $K^*\rho\in\sum_B$ since K is positive and preserves the identity; hence

$$\rho(Kb)=K^*\rho(b)\in \textstyle\sum_B(b)=\bar{W}(b),$$

which implies

VI. *If K is a generalized channel and b an element of the output of K, then*

(7) $$\bar{W}(Kb)\subset \bar{W}(b),$$

that is, K contracts the closed numerical range.

On the other hand, $\sigma(a)$ is contained in $\bar{W}(a)$ and $\bar{W}(b)=\mathrm{co}\,\sigma(b)$ by the hypothesis, so that

$$\sigma(Kb)\subset \bar{W}(Kb)\subset \bar{W}(b)=\mathrm{co}\,\sigma(b),$$

which proves V.

V and VI show that generalized channels have an averaging property. It is also remarked that the proofs of V and VI are essentially C^*-algebraic.

If the output of K is abelian, then every element of the output is normal and consequently convexoid; hence V implies

VII. *If the output* B *of a generalized channel* K *is abelian, then* (5) *is true for any* $b \in B$.

References

[1] S. K. Berberian: Trace and the convex hull of the spectrum in a von Neumann algebra of finite class. Proc. Amer. Math. Soc., **23**, 211–212 (1969).

[2] S. K. Berberian and G. H. Orland: On the closure of the numerical range of an operator. Proc. Amer. Math. Soc., **18**, 499–503 (1967).

[3] C. Davis: Operator-valued entropy of a quantum mechanical measurement. Proc. Japan Acad., **37**, 533–538 (1961).

[4] J. Dixmier: Les algèbres d'opérateurs dans l'espace Hilbertien. Gauthier-Villars, Paris (1957).

[5] M. Echigo (Choda) and M. Nakamura: A remark on the concept of channels. Proc. Japan Acad., **38**, 307–309 (1962).

[6] M. Nakamura and H. Umegaki: On von Neumann's theory of measurements in quantum statistics. Math. Japon., **7**, 151–157 (1962).

[7] J. von Neumann: Mathematische Grundlagen der Quantenmechanik. Springer, Berlin (1932).

[8] I. E. Segal: A class of operator algebras which are determined by groups. Duke Math. J., **18**, 221–265 (1951).

[9] J. G. Stampfli and J. P. Williams: Growth conditions and numerical range in a Banach algebra. Tohoku Math. J., **20**, 417–424 (1968).

[10] Z. Takeda: On the representation of operator algebras. Proc. Japan Acad., **30**, 299–304 (1954).

[11] H. Umegaki: Conditional expectation in operator algebra. Tohoku Math. J., **6**, 177–181 (1954).