35. *Surgery and Singularities in Codimension Two*

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1. Statement of results. Throughout this paper, W^{m+2} denotes a compact connected 1-connected PL $m+2$-manifold which is a Poincaré complex of formal dimension m. A closed PL submanifold L^m of W^{m+2} with codimension 2 is called a *homotopy spine* if the inclusion map $i: L^m \to W^{m+2}$ is a homotopy equivalence. In this paper, we shall formulate an obstruction theory to finding *locally flat* homotopy spines of W^{m+2}. The problem has been solved in odd dimensional case [1]. Here we shall consider the case where m is even: $m=2n \geqq 6$. An additional condition (H) on W^{2n+2} is also assumed, which is a generalization of simplicity condition for knots [3].

(H) *There exist an S^1-fibration $\xi \xrightarrow{p} W$ and a map $\phi: \partial W^{(n)} \to \xi$, where $\partial W^{(n)}$ is the n-skeleton of some triangulation of ∂W, such that* (i) *ϕ is n-connected and* (ii) *the diagram* $\partial W^{(n)} \xrightarrow{\phi} \xi$, $\partial W^{(n)} \to W \xleftarrow{p} \xi$ *is homotopically commutative.*

Note that $\pi_1(\partial W) \cong \pi_1(\xi)$ is a cyclic group. Denote this group in a multiplicative way by $J_q = \{t^m \mid m \in Z\}/(t^q)$, $q=0,1,2,\cdots$. In § 3, a covariant functor $P_{2n}(*)$ from the category {cyclic groups, onto homomorphisms} to the category {abelian groups, onto homomorphisms} is algebraically introduced. Our results are the following:

Theorem 1.1. *W^{2n+2} admits a locally flat homotopy spine if and only if a well defined obstruction element $\eta(W) \in P_{2n}(\pi_1 \partial W)$ is equal to zero.*

The groups $P_{2n}(J_q)$ have some interesting properties.

Theorem 1.2. (i) $P_{2n}(J_0) \cong C_{2n-1}$ *(Levine's knot cobordism group of $(2n-1, 2n+1)$-knots* [3]*), where J_0 is an infinite cyclic group.* (ii) $P_{2n}(1) \cong P_{2n}$*(Kervaire-Milnor's surgery obstruction group* [2]*), where 1 stands for a trivial group.* (iii) $P_{2n+4}(J_q) = P_{2n}(J_q)$.

A submanifold L^{2n} is said to be 1-*flat* if it is locally flat except at a finite set of points. The obstruction $\eta(W)$ can be described in terms of singularities of 1-flat homotopy spines. We have proved in [1] that W^{2n+2} admits a 1-flat homotopy spine L^{2n}. Define the *singularity* at $p \in L$ by a $(2n-1, 2n+1)$-knot $\sigma_p(L) = (Lk(p, L), Lk(p, W))$. The *total singularity* of L^{2n} in W is the summation $\sigma(L) = \sum_{p \in L} \sigma_p(L)$ in Levine's knot cobordism group. If L^{2n} is 1-flat, this is in fact a finite sum. Let $j_q: C_{2n-1} \to P_{2n}(J_q)$ be the epimorphism induced by the projection $J_0 \to J_q$ (cf. Theorem (1.2), (i)).

Theorem 1.3. *Let L^{2n} be a 1-flat homotopy spine. If $\pi_1(\partial W) \cong \pi_1(W-L)$ $(\cong J_q)$, then $\eta(W) = j_q(\sigma(L)) = \sum_{p \in L} j_q \sigma_p(L) \in P_{2n}(\pi_1 \partial W)$. In particular, $j_q(\sigma(L))$ does not depend on the choice of L^{2n}.*

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Detailed proof will appear elsewhere.

2. Intersectional property of $\pi_{n+1}(E, \partial N)$. Let L^{2n} be a locally flat closed submanifold of W^{2n+2} which represents a generator of $H_{2n}(W^{2n+2}; Z) \cong Z$. Let N be a regular neighbourhood of L, E the exterior of N, i.e., $E = \overline{W-N}$. The frontier $\partial N = N \cap E$ is the total space of an S^1-bundle $\tilde{\omega}: \partial N \to L$. L^{2n} is said to be *exterior k-connected* if $\pi_i(E, \partial N) = 0$ for $i \leqq k$. It is proved in [1] that any locally flat closed submanifold M^{2n} of W^{2n+2} is L-equivalent to L^{2n} which is exterior n-connected. Thus we may suppose that L^{2n} is already exterior n-connected. If $n \geqq 3$, then $\pi_1(\partial N) \cong \pi_1(E) \cong \pi_1(\partial W)$.

Let $f, g: (D^{n+1}, S^n) \to (E^{2n+2}, \partial N)$ be pathed generic immersions such that the restrictions $f|S^n, g|S^n$ are embeddings. Moreover suppose that the compositions $\tilde{\omega} \circ (f|S^n)$, $\tilde{\omega} \circ (g|S^n): S^n \to L^{2n}$ are generic immersions. They are assumed to intersect in general position. We will define $\alpha(f, g), \beta(f, g) \in Z[\pi_1 \partial W]$ as follows. Let $p \in \tilde{\omega} f(S^n) \cap \tilde{\omega} g(S^n)$. ε_p is the sign ± 1 of the intersection at p of $\tilde{\omega} f(S^n)$ and $\tilde{\omega} g(S^n)$ in L^{2n}. $g_p \in \pi_1(\partial N) = \pi_1(\partial W)$ is defined by $g_p = \{*(\text{base point in } \partial N) \xrightarrow{\gamma(f)} p_f \to (\text{along the } S^1\text{-fibre } \tilde{\omega}^{-1}(p) \text{ in the positive direction}) \to p_g \xrightarrow{\gamma(g)^{-1}} *\}$, where $\gamma(f)$ (or $\gamma(g)$) is the path of f (or g) and p_f (or p_g) is the point of $f(S^n)$ (or $g(S^n)$) over p, i.e., $\{p_f\} = f(S^n) \cap \tilde{\omega}^{-1}(p)$ (or $\{p_g\} = g(S^n) \cap \tilde{\omega}^{-1}(p)$). The positive direction of $\tilde{\omega}^{-1}(p)$ is defined by $[L^{2n}] \times [\tilde{\omega}^{-1}(p)] = [\partial N]$, where $\times$ is homology cross product. $\alpha(f, g)$ is the summation $\sum_p \varepsilon_p g_p$ over all $p \in \tilde{\omega} f(S^n) \cap \tilde{\omega} g(S^n)$.

Now the definition of $\beta(f, g)$ is as follows: Let $q \in f(D^{n+1}) \cap g(D^{n+1})$. ε_q is the sign ± 1 of the intersection at q of $f(D^{n+1})$ and $g(D^{n+1})$ in E^{2n+2}. The orientation of E is $[E] = [\partial N] \times$ (the inward direction of E) and the orientation of D^{n+1} is $[D^{n+1}] = [S^n] \times$ (the radial inward direction of D^{n+1}). $g_q \in \pi_1(E) = \pi_1(\partial W)$ is defined by $g_q = \{* \xrightarrow{\gamma(f)} q \xrightarrow{\gamma(g)^{-1}} *\}$. Define $\beta(f, g) = \sum_q \varepsilon_q g_q$ over all $q \in f(D^{n+1}) \cap g(D^{n+1})$. Denote the group ring $Z[\pi_1 \partial W]$ by Λ. A pairing $\lambda(f, g)$ is defined by the following:

$$\lambda(f, g) = \alpha(f, g) + (-1)^{n+1}(1-t) \cdot \beta(f, g) \in \Lambda,$$

where $t \in \pi_1(\partial N)=\pi_1(\partial W)$ is the image of the positive generator of $\pi_1(S^1)$ under the homomorphism $\pi_1(S^1)\to\pi_1(\partial N)$ induced by the inclusion of an S^1-fibre.

Note that neither α nor β are regular homotopy invariant. But we have

Proposition 2.1. *$\lambda(f,g)$ is homotopy invariant and λ defines a map $\pi_{n+1}(E,\partial N)\times\pi_{n+1}(E,\partial N)\to\Lambda$.*

It is easy to see that $\lambda(f,g)=(-1)^n\overline{\lambda(g,f)}\cdot t$, where $-:\Lambda\to\Lambda$ is defined by $\overline{\sum m_it^i}=\sum m_it^{-i}$. Define $V_n^t=\Lambda/\{a-(-1)^n\bar{a}\cdot t\mid a\in\Lambda\}$.

Let $f:(D^{n+1},S^n)\to(E^{2n+2},\partial N)$ be a generic immersion. We may define $\alpha(f)\in V_n^t$ and $\beta(f)\in V_{n+1}=\Lambda/\{a-(-1)^{n+1}\bar{a}\mid a\in\Lambda\}$ (Wall's notation [5]) in a similar manner as we defined $\alpha(f,g)$ and $\beta(f,g)$. We have only to replace intersection points by self-intersections. Multiplication induces $(1-t):V_{n+1}\to V_n^t$, thus $(1-t)\cdot\beta(f)\in V_n^t$.

Let v be a positive "tangent vector field" over ∂N along the S^1-fibres. Suppose that $f(S^n)\subset\partial N$ is transversal to each S^1-fibre, then the restriction $v\mid f(S^n)$ is a never-zero normal field over $f(S^n)$. Let $\mathcal{O}(f)\in \boldsymbol{Z}=\pi_n(S^n)$ be the obstruction to extend the field $v\mid f(S^n)$ to a never-zero normal field over the whole of D^{n+1}. ($\mathcal{O}(f)$ is meaningful even in PL case.) Define $\mu(f)\in V_n^t$ as follows:

$$\mu(f)=\alpha(f)+(-1)^{n+1}(1-t)\cdot\beta(f)+(-1)^{n+1}\mathcal{O}(f).$$

Proposition 2.2. *$\mu(f)$ depends only on the homotopy class of f, and the map $\mu:\pi_{n+1}(E,\partial N)\to V_n^t$ is well defined.*

Let $G=\pi_{n+1}(E,\partial N)$. The triple (G,λ,μ) has analogous properties to a special Hermitian form defined by Wall [5].

Proposition 2.3.

(i) *Under the condition* (H) *of* §1, *G is a finitely generated stably free Λ-module.*

(ii) $\lambda(f,g)=(-1)^n\overline{\lambda(g,f)}\cdot t$.

(iii) *For any $f\in G$, $\lambda(*,f):G\to\Lambda$ is a Λ-homomorphism.*

(iv) $\mu(f+g)=\mu(f)+\mu(g)+\lambda(f,g)$.

(v) $\lambda(f,f)=\mu(f)+(-1)^n\overline{\mu(f)}\cdot t$.

(vi) $\mu(af)=a\mu(f)\bar{a}$ for $a\in\Lambda$.

(vii) *If $n\geqq 3$, f can be represented by a normally embedded $(n+1)$-handle if and only if $\mu(f)=0$* ([1]).

In contrast to Wall's forms, our λ is not generally unimodular. Our form is related to a special Hermitian form over $\boldsymbol{Z}$ by the following.

Proposition 2.4. *Let $G'=G\otimes_\Lambda \boldsymbol{Z}$, $\lambda'=\tilde{\omega}_*\circ\lambda$ and $\mu'=\overline{\tilde{\omega}}_*\circ\mu$, where $\tilde{\omega}_*:\Lambda\to\boldsymbol{Z}$ and $\overline{\tilde{\omega}}_*:V_n^t\to\boldsymbol{Z}/\{m-(-1)^nm\mid m\in\boldsymbol{Z}\}$ are induced by the projection $\tilde{\omega}:\partial N\to L$. Then (G',λ',μ') is a special Hermitian $(-1)^n$-form over $\boldsymbol{Z}$.*

3. *Functor $P_{2n}(*)$.* Let $\Lambda=\boldsymbol{Z}[J_q]$. A triple $X=(G,\lambda,\mu)$ consisting of a Λ-module G and maps $\lambda: G\times G\to\Lambda$ and $\mu: G\to V_n^t$ will be called a *Seifert form* if it satisfies (i)~(iv) of Proposition 2.3 and the statement of Proposition 2.4.

Definition 3.1. A Seifert form X is said to be *null-cobordant* if (i) there exists a sub Λ-module $H\subset G$ such that $\lambda(H\times H)=0$ and $\mu(H)=0$, (ii) H is mapped by the projection $G\to G\otimes_\Lambda \boldsymbol{Z}$ onto a sub-kernel H' of $G'=G\otimes_\Lambda \boldsymbol{Z}$. (Hence, G' is a kernel in the sense of Wall.)

For two Seifert forms X_1, X_2, their *direct sum* $X_1\oplus X_2$ is a triple $(G_1\oplus G_2, \lambda_1+\lambda_2, \mu_1+\mu_2)$. The *inverse* $(-X)$ is $(G, -\lambda, -\mu)$. Let A_{2n} be the Grothendieck group generated by all isomorphism classes of Seifert forms, B_{2n} the subgroup generated by all null-cobordant forms. Define an abelian group $P_{2n}(J_q)$ by the quotient A_{2n}/B_{2n}. Note that $X=(G, \lambda, \mu)$ represents zero element in $P_{2n}(J_q)$ if and only if there exists a null-cobordant form Y such that the direct sum $X\oplus Y$ is null-cobordant.

The obstruction element $\eta(W)$ in Theorem 1.1 is defined by the triple $(\pi_{n+1}(E, \partial N), \lambda, \mu)$.

Theorem 1.2 (i) follows from the fact that any Seifert form (G, λ, μ) over $\boldsymbol{Z}[J_0]$ is represented by $(\pi_{n+1}(E, \partial N\cap E), \lambda, \mu)$ such that $W^{2n+2}=E\cup N$ is a $2n+2$ disk and L^{2n} is an exterior n-connected locally flat proper submanifold of W^{2n+2} with $\partial L^{2n}\cong S^{2n-1}$. Other statements of Theorem 1.2 are immediate consequences of the definition of $P_{2n}(J_q)$.

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