59. *Invariancy of Plancherel Measure under the Operation of Kronecker Product*

By Nobuhiko TATSUUMA
Department of Mathematics, Kyôto University

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1971)

1. Let G be a unimodular locally compact group of type I. For such a group, so-called Plancherel formula was given by F. I. Mautner [2], I. E. Segal [3], and H. Sunouchi [4], as follows.

Consider the dual $\hat{G}$ (the set of all equivalence classes of irreducible unitary representations) of G, and put $U_f(\omega)=\int_G f(g)U_g(\omega)dg$ for any function f in $L^1(G)$ and any unitary representation $\omega=\{\mathfrak{H}(\omega), U_g(\omega)\}$ of G. Then, there exists a measure μ (Plancherel measure) over $\hat{G}$, such that for any function f in $L^1(G)\cap L^2(G)$, the equation (1) is valid.

$$\|f\|^2=\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega). \tag{1}$$

Here $|||U_f(\omega)|||$ is the Hilbert-Schmidt norm of the operator $U_f(\omega)$.

This formula is considered as an extension of the Plancherel formula for abelian locally compact groups. But in this abelian case, $\hat{G}$ becomes an abelian locally compact group too, and the Plancherel measure μ is just invariant measure over $\hat{G}$.

The group operation of $\hat{G}$ is given by the ordinary product of characters as functions on G, that is, the Kronecker product of 1-dimensional representation. So the invariancy of Plancherel measure is that,

$$d\mu(\chi_0\otimes\chi)=d\mu(\chi), \qquad \text{for any } \chi_0 \text{ in } \hat{G}, \tag{2}$$

and this is equivalent to,

$$\int_{\hat{G}} |\tilde{f}(\chi_0\otimes\chi)|^2 d\mu(\chi)=\int_{\hat{G}} |\tilde{f}(\chi)|^2\, d\mu(\chi), \tag{3}$$
$$\text{for any } \chi_0 \text{ in } \hat{G} \text{ and } f \text{ in } L^1(G)\cap L^2(G).$$

Here $\tilde{f}$ shows the Fourier transform of f.

In general case, an analogue of (3) may be constructed as follows. At first, by virtue of (1), we replace Fourier transform $\tilde{f}$ of function f by the operator-valued function $U_f(\omega)$, then the term $|\tilde{f}(\chi_0\otimes\chi)|^2$ is replaced by $|||U_f(\omega_0\otimes\omega)|||^2$.

On the other hand, the well-known relation $\omega_0\otimes\mathfrak{R}\sim\sum_{\dim\omega_0}\oplus\mathfrak{R}$, for the regular representation $\mathfrak{R}$ and any representation ω_0, suggests that, in general form, the factor $(\dim\omega_0)^{-1}$ is needed in the left hand side. So, one of the purposes of this paper is to show the equation (4) for finite dimensional representation ω_0.

$$(\dim \omega_0)^{-1}\int_{\hat{G}} |||U_f(\omega_0\otimes\omega)|||^2\, d\mu(\omega)=\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega). \tag{4}$$

For the case when ω_0 is infinite dimensional the left hand side of (4) is meaningless, so we have to take some modification.

The definition of the Hilbert-Schmidt norm gives,

$$|||U_f(\omega_0\otimes\omega)|||^2=\sum_j \sum_k \|U_f(\omega_0\otimes\omega)(v_j(\omega_0)\otimes v_k(\omega))\|^2. \tag{5}$$

Here $\{v_j(\omega_0)\}$ and $\{v_k(\omega)\}$ are any orthonormal basis in $\mathfrak{H}(\omega_0)$ and $\mathfrak{H}(\omega)$ respectively.

For fixed basis $\{v_j(\omega_0)\}$ of $\mathfrak{H}(\omega_0)$, we take a partial sum of (5) with respect to j, and put

$$\phi_N(\omega)\equiv\frac{1}{N}\sum_j^N (\sum_k \|U_f(\omega_0\otimes\omega)(v_j(\omega_0)\otimes v_k(\omega))\|^2), \tag{6}$$

then our required equation is

$$\lim_{N\to\infty}\int_{\hat{G}} \phi_N(\omega)\, d\mu(\omega)=\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega). \tag{7}$$

But, in this paper, we get the stronger result as follows.

Theorem. *For any $v(\omega_0)$ in $\mathfrak{H}(\omega_0)$, such that $\|v(\omega_0)\|=1$, the equation (8) is valid.*

$$\int_{\hat{G}} \sum_k \|U_f(\omega_0\otimes\omega)(v(\omega_0)\otimes v_k(\omega))\|^2\, d\mu(\omega)=\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega), \quad \text{for any } f \text{ in } L^1(G)\cap L^2(G). \tag{8}$$

Evidently (4) and (7) are immediate results of (8).

Lastly we shall give an example, for which the limiting process in (7) can't enter under the integral sign, i.e.,

$$\int_{\hat{G}} \lim_{N\to\infty} \phi_N(\omega)\, d\mu(\omega)\neq\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega). \tag{9}$$

2. Proof of the theorem. The proof is given by direct calculations. We take $v(\omega_0)$, $\{v_k(\omega)\}$, f, as in the theorem, and an orthonormal basis $\{v_l(\omega_0)\}$ in $\mathfrak{H}(\omega_0)$.

$$\begin{aligned}
I&=\int_{\hat{G}} \sum_k \|U_f(\omega_0\otimes\omega)(v(\omega_0)\otimes v_k(\omega))\|^2\, d\mu(\omega)\\
&=\int_{\hat{G}} \sum_k \left\|\int_G f(g)(U_g(\omega_0)v(\omega_0)\otimes U_g(\omega)v_k(\omega))dg\right\|^2 d\mu(\omega)\\
&=\int_{\hat{G}} \sum_k \left\{\int_G\int_G f(g_1)\overline{f(g_2)}\langle U_{g_1}(\omega_0)v(\omega_0),\, U_{g_2}(\omega_0)v(\omega_0)\rangle\right.\\
&\qquad \left.\times\langle U_{g_1}(\omega)v_k(\omega),\, U_{g_2}(\omega)v_k(\omega)\rangle dg_1dg_2\right\} d\mu(\omega)\\
&=\int_{\hat{G}} \sum_k \left\{\int_G\int_G f(g_1)\overline{f(g_2)}\sum_l \langle U_{g_1}(\omega_0)v(\omega_0),\, v_l(\omega_0)\rangle\right.\\
&\qquad \left.\times\overline{\langle U_{g_2}(\omega_0)v(\omega_0),\, v_l(\omega_0)\rangle}\langle U_{g_1}(\omega)v_k(\omega),\, U_{g_2}(\omega)v_k(\omega)\rangle dg_1dg_2\right\} d\mu(\omega).
\end{aligned}$$

But in the right hand side, the absolute value of the integrand is bounded by

$$\int_G \int_G |f(g_1)||f(g_2)| \sum_l |\langle U_{g_1}(\omega_0)v(\omega_0), v_l(\omega_0)\rangle| |\langle U_{g_2}(\omega_0)v(\omega_0), v_l(\omega_0)\rangle|$$
$$\times |\langle U_{g_1}(\omega)v_k(\omega), U_{g_2}(\omega)v_k(\omega)\rangle| \, dg_1 dg_2$$
$$\leqq \int_G \int_G |f(g_1)||f(g_2)| \Big(\sum_l |\langle U_{g_1}(\omega_0)v(\omega_0), v_l(\omega_0)\rangle|^2\Big)^{1/2}$$
$$\times \Big(\sum_l |\langle U_{g_2}(\omega_0)v(\omega_0), v_l(\omega_0)\rangle|^2\Big)^{1/2} \| U_{g_1}(\omega)v_k(\omega)\| \, \| U_{g_2}(\omega)v_k(\omega)\| \, dg_1 dg_2$$
$$\leqq \left\{ \int_G |f(g)| \, \| U_g(\omega_0)v(\omega_0)\| \, \| v_k(\omega)\| \, dg_1 \right\}^2 = \left\{ \int_G |f(g)| \, dg \right\}^2.$$

So by the Fubini's theorem, we can take the sum by l before the integrals by g_1, g_2.

$$I = \int_{\hat{G}} \sum_k \sum_l \int_G \int_G f(g_1)\overline{f(g_2)} \langle U_{g_1}(\omega_0)v(\omega_0), v_l(\omega_0)\rangle \overline{\langle U_{g_2}(\omega_0)v(\omega_0), v_l(\omega_0)\rangle}$$
$$\times \langle U_{g_1}(\omega)v_k(\omega), U_{g_2}(\omega)v_k(\omega)\rangle dg_1 dg_2 \, d\mu(\omega)$$
$$= \int_{\hat{G}} \sum_k \sum_l \left\| \int_G f(g) \langle U_g(\omega_0)v(\omega_0), v_l(\omega_0)\rangle U_g(\omega)v_k(\omega) \, dg \right\|^2 d\mu(\omega)$$
$$= \sum_l \int_{\hat{G}} \sum_k \| U_{f\times u_l}(\omega)v_k(\omega)\|^2 \, d\mu(\omega).$$

Here $u_l(g) \equiv \langle U_g(\omega_0)v(\omega_0), v_l(\omega_0)\rangle$.

$$I = \sum_l \int_G ||| U_{f\times u_l}(\omega)|||^2 \, d\mu(\omega) = \sum_l \int_G |f(g)u_l(g)|^2 \, dg$$
$$= \int_G \sum_l |f(g)|^2 |\langle U_g(\omega_0)v(\omega_0), v_l(\omega_0)\rangle|^2 \, dg$$
$$= \int_G |f(g)|^2 \, \| U_g(\omega_0)v(\omega_0)\|^2 \, dg = \int_G |f(g)|^2 \, \| v(\omega_0)\|^2 \, dg$$
$$= \int_G |f(g)|^2 \, dg = \int_{\hat{G}} ||| U_f(\omega)|||^2 \, d\mu(\omega).$$

That is, the equation (8) is proved.

Corollary. *If* $\dim \omega_0 < +\infty$, *then,*

$$(\dim \omega_0)^{-1} \int_{\hat{G}} ||| U_f(\omega_0 \otimes \omega)|||^2 \, d\mu(\omega) = \int_{\hat{G}} ||| U_f(\omega)|||^2 \, d\mu(\omega). \tag{10}$$

for any f *in* $L^1(G) \cap L^2(G)$.

3. Example. Let G be the real unimodular group of second order. Now we shall construct f, ω_0 on G for which the inequality (9) is valid. We quote the notations in the previous paper [5].

At first, we fix the positive integer (or half-integer) m ($\geqq 3/2$), and the normalized highest vector v_m in $\mathfrak{H}(D_m^-)$, that is, v_m is determined up to constant factor as the vector satisfying

$$F^+(D_m^-)v_m = 0. \tag{11}$$

Put

$$f(g) \equiv \overline{\langle U_g(D_m^-)v_m, v_m\rangle}, \tag{12}$$

then the V. Bargmann's results ([1]) and calculations of eigenvalue for Laplacian show the followings,

(a) $f(g)$ is in $L^1(G) \cap L^2(G)$.

(b) For given irredudicble representation ω and its canonical basis $\{\zeta_k(\omega)\}$ (cf. [5] p. 318),

$$\begin{aligned}\langle U_f(\omega)\zeta_k(\omega), \zeta_l(\omega)\rangle &= \int_G \overline{\langle U_g(D_m^-)v_m, v_m\rangle}\langle U_g(\omega)\zeta_k(\omega), \zeta_l(\omega)\rangle dg \\ &= (2m-1)^{-1}\,\delta_{m,-k}\,\delta_{m,-l}, \qquad \text{for } \omega = D_m^-, \\ &= 0, \qquad \text{for } \omega = D_n^-(n \neq m), D_n^+, C_l^t, I.\end{aligned} \tag{13}$$

(b) shows that,

$$U_f(\omega)v(\omega) = 0, \qquad \text{for } \omega \neq D_m^-, \tag{14}$$

$$U_f(D_m^-)v = (2m-1)^{-1}\langle v, v_m\rangle v_m, \qquad \text{for } v \in \mathfrak{H}(D_m^-). \tag{15}$$

That is,

$$\begin{aligned}|||U_f(\omega)|||^2 &= (2m-1)^{-2}, \qquad \text{for } \omega = D_m^-, \\ &= 0, \qquad \text{otherwise}.\end{aligned} \tag{16}$$

From the definition of the Hilbert-Schmidt norm, it is easy to see that

$$|||U_f(\omega_0 \otimes \omega)|||^2 = d|||U_f(D_m^-)|||^2 = d(2m-1)^{-2}.$$

Here d is the multiplicity of D_m^--components in the representaiton $\omega_0 \otimes \omega$.

On the other hand, we can deduce the following by just similar arguments as the proof of Proposition 1 in [5].

Lemma. *For fixed s (positive integer or half-integer), $D_s^+ \otimes \omega$ contains D_m^- once time only when $\omega = D_n^-(n \geqq s+m$, and $m+n+s$; integer). And for the other irreducible ω, $D_s^+ \otimes \omega$ does not contain D_m^-.*

This lemma determines the value of the function,

$$\begin{aligned}|||U_f(D_s^+ \otimes \omega)|||^2 &= (2m-1)^{-2}, \qquad \text{for } \omega = D_n^-(n \geqq s+m, m+n+s\text{; integer}), \\ &= 0, \qquad \text{otherwise}.\end{aligned}$$

That is, for $\omega_0 = D_s^+$,

$$\begin{aligned}\lim_{N\to\infty} \phi_N(\omega) &= \lim_{N\to\infty} \frac{1}{N} \sum_j^N \sum_k^\infty \|U_f(D_s^+ \otimes \omega)(\zeta_j^s \otimes \zeta_k(\omega))\|^2 \\ &\leqq \lim_{N\to\infty} \frac{1}{N} \sum_j^\infty \sum_k^\infty \|U_f(D_s^+ \otimes \omega)(\zeta_j^s \otimes \zeta_k(\omega))\|^2 \\ &= \lim_{N\to\infty} \frac{1}{N} |||U_f(D_s^+ \otimes \omega)|||^2 \equiv 0,\end{aligned}$$

So that,

$$\int_{\hat{G}} \lim_{N\to\infty} \phi_N(\omega)\, d\mu(\omega) = 0,$$

$$\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega) = \int_G |f(g)|^2\, dg = (2m-1)^{-1} \neq 0.$$

References

[1] V. Bargmann: Irreducible unitary representations of the Lorentz group. Ann. of Math., **48**, 568–640 (1947).

[2] F. I. Mautner: Unitary representations of locally compact groups. I: Ann. of Math., **51**, 1–25 (1950), II: Ann. of Math., **52**, 528–556 (1950).

[3] I. E. Segal: An extension of Plancherel's formula to separable unimodular groups. Ann. of Math., **52**, 272–292 (1950).
[4] H. Sunouchi: An extension of Plancherel formula to unimodular groups. Tôhoku Math. J., **4**, 216–230 (1952).
[5] N. Tatsuuma: A duality theorem for the real unimodular group of second order. J. Math. Soc. Japan, **17**, 313–332 (1965).