116. *Modules over Bounded Dedekind Prime Rings. I*

By Hidetoshi MARUBAYASHI
College of General Education, Osaka University
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The purpose of this paper is to generalize the theory of modules over commutative Dedekind rings [3] to the case of modules over bounded Dedekind prime rings.

1. Definitions and notations. In this paper, all rings have identity and are associative, and modules are unitary. Ideals always mean two-sided ideals. Let R be a prime Goldie ring and let Q be the quotient ring of R. Then R is called a *Dedekind ring* if R is a maximal order in Q and every right (left) R-ideal is projective (see [8]). R is *bounded* if every integral one-sided R-ideal contains a non-zero ideal. Let M be an R-module. We say that $m \in M$ is a *torsion element* if there is a regular element c in R such that $mc=0$. Since R satisfies the Ore condition, the set of torsion elements of M is a submodule $T \subseteq M$. And M/T is evidently torsion-free (has no torsion elements). Let x be an element of M. Then we define $O(x)=\{r \in R \mid xr=0\}$ and say that $O(x)$ is an *order right ideal* of x. Let P be a prime ideal of R and let M be a torsion R-module. Then we say that M is *primary* (P-primary) if $O(x)$ contains a power of P for every element x in M. A submodule S of an R-module is said to be *pure* if $Sc=S \cap Mc$ for every regular element c in R. In particular, S is said to be *strongly pure* if $Sr=S \cap Mr$ for every element r in R. Then the following properties hold: (i) Any direct summand is strongly pure. (ii) A (strongly) pure submodule of a (strongly) pure submodule is (strongly) pure. (iii) The torsion submodule is pure. (iv) If M/S is torsion-free, then S is pure. We define an R-module M to be *divisible* if $Mc=M$ for all regular element c in R. Finally J or $J(R)$ always denotes the Jacobson radical of the ring R. The ring R is *local* if R/J is artinian and $\bigcap_{s=1}^{\infty} J^s=(0)$. R is *s-local* if R is local and R/J is a division ring.

2. Modules over bounded Dedekind prime rings. Let R be a semi-hereditary prime Goldie ring, let Q be the quotient ring of R and let M be a finitely generated torsion-free R-module. Then the sequence $0 \to M \to M \otimes_R Q$ is exact and $M \otimes_R Q$ is Q-projective. So $M \otimes_R Q$ is a submodule of a finitely generated free Q-module. Furthermore, since M is finitely generated, M is a submodule of a free R-module. Hence M is R-projective. Now let u be a uniform element of R. Then the short exact sequence $0 \to O(u) \to R \to uR \to 0$ splits. So R is a direct sum of a finite number of uniform right ideals. Hence we have

Theorem 1. *Let R be a semi-hereditary prime Goldie ring and let M be a finitely generated R-module with torsion submodule T. Then*

(i) *M/T is a projective R-module and is a direct sum of a finite number of uniform right ideals.*

(ii) $M=T\oplus M/T$.

From now on, R will be a bounded Dedekind prime ring and let Q be the simple artinian quotient ring of R. Since every integral right R-ideal contains non-zero ideals, we have

Theorem 2. *Any torsion module over a bounded Dedekind prime ring is a direct sum of primary submodules.*

Let P be a prime ideal of R and let R_P be the local ring of R with respect to P in the sense of Goldie [2]. Then $R_P=\{ac^{-1} \mid a \in R, c \in \mathcal{C}(P)\}$ by Lemma 2.10 of [7], where $\mathcal{C}(P)=\{c \in R \mid cx \in P \Rightarrow x \in P\}$. Now, let M be a P-primary R-module. Then we can regard, in a natural way, M as an R_P-module.

Lemma 1. *Let M be any module, let S be a submodule such that M/S is a direct sum of modules U_i, and let T_i be the inverse image in M of U_i. Suppose S is a direct summand of each T_i. Then S is a direct summand of M.*

Lemma 2. *Let R be a bounded Dedekind prime ring, let M be an R-module and let S be a pure submodule such that M/S is torsion. If x_0 is an element of M/S, then there exists an element x in M, which maps on x_0 mod S, and $O(x)=O(x_0)$.*

We shall call an R-module *decomposable* if it is a direct sum of cyclic modules and uniform right ideals.

From Lemma 1 and Lemma 2 we have

Theorem 3. *Let R be a bounded Dedekind prime ring, let M be an R-module, and let S be a pure submodule such that M/S is decomposable. Then S is a direct summand of M.*

By Theorems 1 and 3, we have

Corollary. *Let R be a bounded Dedekind prime ring, let M be a finitely generated R-module and let S be a submodule. Then the following three conditions are equivalent:*

(i) *S is a direct summand of M.*

(ii) *S is a strongly pure submodule of M.*

(iii) *S is a pure submodule of M.*

Since every proper homomorphic image of a bounded prime Dedekind ring is generalized uniserial, by Theorem 2.54 of [1; p. 79], we have

Theorem 4. *Let R be a bounded Dedekind prime ring and let M*

be an R-module of bounded order (*i.e.*, $Mc=0$ *for some regular element* c *of* R). *Then* M *is a direct sum of cyclic modules, each of which is an artinian module.*

Theorem 5. *Let* R *be a bounded Dedekind prime ring, let* M *be an* R-*module, and let* S *be a strongly pure submodule of bounded order. Then* S *is a direct summand of* M.

Corollary. *Let* D *be a bounded Dedekind domain, let* M *be a* D-*module, and let* S *be a pure submodule of bounded order. Then* S *is a direct summand of* M.

Theorem 6. *Let* R *be a bounded Dedekind prime ring and let* M *be an* R-*module such that* M/T *is finitely generated, where* T *is the torsion submodule of* M. *If* S *is a pure submodule of bounded order, then* S *is a direct summand of* M.

Let R_P be the local ring of R with respect to P. Then $R_P=(D)_k$, where D is a bounded s-local domain in which every one-sided ideal of D is an ideal and every ideal of D is a power of $J(D)$. Furthermore, we let $J(D)=p_0D=Dp_0$ for some $p_0\in D$. Then $J(R_P)=p_0R_P=R_Pp_0$. Now an idempotent e in R_P is called *uniform* if eR_P is a uniform right ideal of R_P. Then the sequence

$$(*)\qquad 0\to eR_P/eP'^n \xrightarrow{\varphi_n} eR_P/eP'^{n+1}$$

is exact, where $P'=J(R_P)$ and $\varphi_n(eq+eP'^n)=ep_0q+eP'^{n+1}$ for every q in R_P.

Lemma 3. *Let* R *be a bounded Dedekind prime ring. Then any simple* R-*module is primary and is isomorphic to* eR/eP *for some prime ideal* P, *where* e *is a uniform idempotent contained in* R_P.

We denote the injective hull of an R-module A by $E(A)$.

Theorem 7. *The inductive limit* E *of the rings* eR_P/eP'^n, $n=1,2,\cdots$, *under the homomorphisms* φ_n *defined in* $(*)$, *is divisible and is isomorphic to* $E(eR/eP)$.

We shall call the module $E(eR/eP)$ in Theorem 7 a *module of type* P^∞.

By Theorem 1.4 of [6], Theorem 3.4 of [5] and Lemma 3 we obtain the following two theorems:

Theorem 8. *Let* R *be a bounded Dedekind prime ring with quotient ring* Q. *Then any divisible* R-*module is the direct sum of minimal right ideals of* Q *and modules of type* P^∞ *for various prime ideals* P.

Theorem 9. *Any module* M *over a bounded Dedekind prime ring possesses a unique largest divisible submodule* D; $M=D\oplus E$, *where* E *has no divisible submodules.*

Let P be a prime ideal of a bounded Dedekind prime ring R. Then we denote the completion of R_P with respect to $J(R_P)$ by $\hat{R}_P$.

Lemma 4. *$\hat{R}_P$ is a bounded local Dedekind prime ring which is a principal ideal ring.*

As is well known, the ring of endomorphisms of a group of type p^∞ is isomorphic to the ring of p-adic integers [4; p. 155]. In our case, we have

Theorem 10. *Let P be a prime ideal of a bounded Dedekind prime ring R and let E be an R-module of type P^∞. Then*

(i) *E is in a natural way an $\hat{R}_P$-module.*

(ii) *E is an $\hat{R}_P$-module of type $\hat{P}^\infty$, where $J(\hat{R}_P)=\hat{P}$.*

(iii) *The ring of endomorphisms of E is isomorphic to $e\hat{R}_P e$, where e is a uniform idempotent in R_P.*

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