22. On the Gaps between the Refinements of the Increasing Open Coverings

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§1. **Preliminaries.** In normal spaces, C. H. Dowker [1] gave the characterizations of countably paracompact spaces. After a time, F. Isikawa [3] discussed about the generalizations of its characterizations.

One of our purposes, in this paper, is to study the gaps, in which their characterizations of countably paracompact spaces need not be identical without the normality. Recently, P. Zenor [7] (resp. S. Sasada [4]) defined the topological class which was contained in the countably paracompact class and was a generalization of the property being stated by F. Isikawa [3] (resp. C. H. Dowker [1]: Theorem 2).

Another purpose of this paper is to find the gaps between the above topological spaces. Before stating properties, we will recall or define the terms which are used in this paper.

Let $\mathfrak{A}=\{A_\alpha \mid \alpha \in A\}$ be a collection of subsets of a topological space X. $\mathfrak{A}$ is said to be *monotone increasing* (resp. *monotone decreasing*) if A is well ordered and $A_\alpha \supseteq A_\beta$ (resp. $A_\alpha \subseteq A_\beta$) for each $\alpha, \beta \in A$ with $\alpha \geqslant \beta$. The space X is said to have a *$\mathfrak{B}$-property* (resp. *a weak $\mathfrak{B}$-property*) if for each monotone decreasing family $\{F_\alpha / \alpha \in A\}$ of closed sets of X with vacuous intersection there is a monotone decreasing family (resp. a simple collection)$\{G_\alpha \mid \alpha \in A\}$ of open sets of X such that $\bigcap_{\alpha \in A} \overline{G}_\alpha$[1] $=\emptyset$ and $G_\alpha \supset F_\alpha$ for each $\alpha \in A$. X is said to have a *countable $\mathfrak{B}$-property* (resp. a *countable weak $\mathfrak{B}$-property*) if the indexed set A of the definition of the $\mathfrak{B}$-property (resp. the weak $\mathfrak{B}$-property) is the countable set. X is a *$\mathfrak{B}$-space* (resp. *a weak $\mathfrak{B}$-space, a countable $\mathfrak{B}$-space, a countable weak $\mathfrak{B}$-space*) if X has the $\mathfrak{A}$-property (resp. the weak $\mathfrak{B}$-property, the countable $\mathfrak{B}$-property, the countable weak $\mathfrak{B}$-property).

There is natural dual characterizations of $\mathfrak{B}$-property etc. in terms of the monotone increasing open covering (see T. Tani and Y. Yasui [6]). The following properties are equivalent: (1) the countable $\mathfrak{B}$-property (2) the countable weak $\mathfrak{B}$-property, (3) the countable paracompactness (see F. Ishikawa [3]).

The normal $\mathfrak{B}$-space (resp. the normal weak $\mathfrak{B}$-space) is said to be

1) $\overline{G}_\alpha$ denotes the closure of G_α.

α_1-space (resp. *α_2-space*) by S. Sasada [4].

§2. Propositions. In normal spaces, the countable paracompactness is equivalent to (*) every countable closed collection $\{F_i \mid i=1,2,\cdots\}$ with the vacuous intersection has the countable open collection $\{G_i \mid i=1,2,\cdots\}$ with the vacuous intersection such that $G_i \supset F_i$ for each $i=1,2,\cdots$ (see C. H. Dowker [1]: Theorem 2). Without the normality, the following property is well known:

Theorem 1 (F. Isikawa [3]). *A space X is countably paracompact if and only if X is the countable $\mathfrak{B}$-space.*

While the countable $\mathfrak{B}$-property is stronger than the property (*), we will show that the converse is not true. The symbol $\underset{\sim}{R}$ (resp. $\underset{\sim}{P}$) will always denote the real numbers (resp. the irrational numbers) with the usual topology.

Lemma. *There exist two uncountable subsets X_1, X_2 of $\underset{\sim}{P}$ having the following properties:*

(1) $X_1 \cap X_2 = \emptyset$

(2) $\overline{X}_i \supset \underset{\sim}{P}$ *for $i=1,2$ (where the closure denotes in $\underset{\sim}{R}$) and*

(3) $G \cap X_i$ *is uncountable for $i=1$, 2 and any nonempty open set G of $\underset{\sim}{R}$.*

Proof. Let $\mathfrak{B}_n=\{(a,b) \mid a, b \in \underset{\sim}{P}, b-a=1/5^n\}$ for each positive integer n, and let $\mathfrak{B}=\bigcup_{i=1}^{\infty}\mathfrak{B}_i$, then it is seen that $\{\mathfrak{B}_i \mid i=1,2,\cdots\}$ is the mutually disjoint collection (where we may suppose that $\mathfrak{B}$ is well ordered and hence $\mathfrak{B}=\{B_\alpha \mid \alpha<\omega_1\}$[2]).

For each $\alpha<\omega_1$, let $B_\alpha=(a_\alpha, b_\alpha)$ for some $a_\alpha, b_\alpha \in \underset{\sim}{P}$. Furthermore let $c_{\omega_0(\alpha)}=(4a_\alpha+b_\alpha)/5$, $d_{\omega_0(\alpha)}=(a_\alpha+4b_\alpha)/5$, $a_{m(\alpha)}=c_{\omega_0(\alpha)}-1/5^{n+1}\cdot 1/(m+1)$ and $b_{m(\alpha)}=d_{\omega_0(\alpha)}+1/5^{n+1}\cdot 1/(m+1)$ for each $m=0,1,2,\cdots$ and $B_\alpha \in \mathfrak{B}_n$. Then we have:

(1) $\lim_{m\to\infty} a_{m(\alpha)}=c_{\omega_0(\alpha)}$ and $\lim_{m\to\infty} b_{m(\alpha)}=d_{\omega_0(\alpha)}$ for each $\alpha<\omega_1$,

(2) $3/5^{n+1}<b_{m(\alpha)}-a_{m(\alpha)}<1/5^n$ for each $m=1,2,3,\cdots$ and $B_\alpha \in \mathfrak{B}_n$,

and (3) if $\alpha \neq \beta$, then $(a_{m(\alpha)}, b_{m(\alpha)}) \neq (a_{t(\beta)}, b_{t(\beta)})$ for each $m, t=0,1,\cdots,\omega_0$[3].

If we consider that $A(\alpha)=\{m(\alpha) \mid 0 \le m \le \omega_0\}$ has the natural order (i.e. $m(\alpha) \le n(\alpha)$ if and only if $m \le n$) for each $\alpha<\omega_1$, then we may assume that $\bigcup_{\alpha<\omega_1} A(\alpha)$ will denote the product set $[0,\omega_0]\times[0,\omega_1)$ with the dictionary order, and hence we may let $\mathfrak{B}'=\{B'_\lambda \mid \lambda \in \bigcup_{\alpha<\omega_1} A(\alpha)\}$ where we let $B'_\lambda=(a_\lambda, b_\lambda)$ for each $\lambda \in \bigcup_{\alpha<\omega_1} A(\alpha)$. From (1), (2) and (3), we will find the subset $\{x_\lambda \mid \lambda \in \bigcup_{\alpha<\omega_1} A(\alpha)\}$ of $\underset{\sim}{P}$ such that x_λ is in $\underset{\sim}{P} \cap (B_\lambda - \{x_\mu \mid \mu<\lambda\})$

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2) A symbol ω_1 denotes the first uncountable ordinal number.

3) A symbol ω_0 denotes the first infinite ordinal number.

for each $\lambda \in \bigcup_{\alpha<\omega_1} A(\alpha)$ (it will be possible from the fact that $\{x_\mu \mid \mu<\lambda\}$ is countable for each $\lambda \in \bigcup_{\alpha<\omega_1} A(\alpha)$). Let $X_1=\{x_\lambda \mid \lambda$: the limit ordinal number of $\bigcup_{\alpha<\omega_1} A(\alpha)\}$ and $X_2=\{x_\lambda \mid \lambda$: not the limit ordinal number of $\bigcup_{\alpha<\omega_1} A(\alpha)\}$, then X_1 and X_2 have been chosen so that our requirement of the lemma is satisfied. It completes the proof of the lemma.

The proof of the following theorem is the same as for the above lemma.

Theorem 2. *If X is an uncountable subset of $\underset{\sim}{R}$ such that $G\cap X$ is uncountable for every non-empty open set G of $\underset{\sim}{R}$, then there exist the uncountable subsets X_1, X_2 of X having the following properties:*

(1) $X=X_1\cup X_2$

(2) $X_1\cap X_2=\emptyset$

(3) *X_i is dense in $\underset{\sim}{R}$ for $i=1,2$*

and (4) *$X_i\cap G$ is uncountable for every non-empty open set of $\underset{\sim}{R}$.*

§3. Examples.

Example 1. Let X be the subset $\{(x,y) \mid y\geqq 0\}$ of $\underset{\sim}{R}\times\underset{\sim}{R}$ with the following neighborhood base $\{S_\varepsilon(p) \mid \varepsilon>0\}$ at $p=(x,y)\in X$:

(1) $S_\varepsilon(p)=\{(u,v)\in X \mid d((u,v),$[4]$p)<\varepsilon\}$ if $y>0$,

(2) $S_\varepsilon(p)=\{(u,v)\in X \mid d((u,v),\ (x,\varepsilon))<\varepsilon\}\cup\{p\}$ if $y=0$.

Then *this completely regular, not normal space X has the property* (*) *but not the countable $\mathfrak{B}$-property, that is, X is not countably paracompact.*

Proof. We put $L=\{(x,y)\in X \mid y=0\}$.

X having the property (*). $\mathfrak{G}=\{G_i \mid i=1,2,\cdots\}$ be any increasing open covering of X. Since $G_i\cap(X-L)$ is an open set of $\underset{\sim}{R}\times\underset{\sim}{R}$, we have the countable increasing closed subsets $\{F^i_j \mid j=1,2,\cdots\}$ of $\underset{\sim}{R}\times\underset{\sim}{R}$ such that $G_i\cap(X-L)=\bigcup_{j=1}^{\infty} F^i_j$. F^i_j being the closed set of $\underset{\sim}{R}\times\underset{\sim}{R}$ and being contained in $(X-L)$, F^i_j is the closed set of X for each i, $j=1,2,\cdots$.

If we let $F_n=\left(\bigcup_{i+j=n+1} F^i_j\right)\cup(G_n\cap L)$ for each $n=1,2,\cdots$, it is easily seen that $\{F_n \mid n\}$ is a closed covering of X and $G_n\supset F_n$ for each $n=1,2,\cdots$. Therefore X has the property (*).

X being not countably paracompact. From Theorem 2, we have the countable subsets $\{B_i \mid i=1,2,\cdots\}$ of L such that

(i) $L=\bigcup_{i=1}^{\infty} B_i$,

(ii) $B_i\cap B_j=\emptyset$ for $i\neq j$,

and (iii) $G\cap B_i$ is uncountable for any nonempty open set G of $\underset{\sim}{R}$, and $i=1,2,\cdots$.

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4) d denotes the usual metric function of $\underset{\sim}{R}\times\underset{\sim}{R}$.

Let $G_n=(X-L)\cup\left(\bigcup_{i=1}^{n} B_i\right)$ for $n=1, 2, \cdots$, then $\{G_n \mid n\}$ is the increasing open covering of X by the property (i). Assume X being countably paracompact, then there exists a countable open covering $\{D_i \mid i\}$ of X such that $G_i \supset \overline{D}_i$ for each $i=1, 2, \cdots$. But, from the properties (ii) and (iii) of $\{B_i \mid i\}$, we can show that $D_i \cap L$ being countable for each $i=1, 2, \cdots$ under the adequate computations. This contradicts the cardinality of L, that is, X is not the countably paracompact space.

Example 2. Let S be the Sorgenfrey line (see R. H. Sorgenfrey [5]), i.e., S consists of the set of real numbers, topologized by taking as a base all half-open intervals of the form $[a, b)$ with $a<b$. While it is well known that *S is the Lindelöf space (and hence $\mathfrak{B}$-space)* (see R. H. Sorgenfrey [5]), *$S\times S$ is not the countably paracompact space (and hence not the $\mathfrak{B}$-space).*

Proof. The fact that $S\times S$ is not the countably paracompact space is seen under the same way of Example 2, that is, instead of the real line L of the proof of Example 2, we consider the line $\{(x, y) \mid (x, y) \in S\times S,\ x+y=1\}$. We omit the proof.

Lastly we will discuss the gap between the $\mathfrak{B}$-property and the weak $\mathfrak{B}$-property.

Example 3. Let $X=[0, \omega_1)$ with the order topology, then *X is normal and the weak $\mathfrak{B}$-space (and hence, the countable $\mathfrak{B}$-space), but not the $\mathfrak{B}$-space.*

Proof. It is well known that X is normal and countably compact, and therefore X is the countable $\mathfrak{B}$-space.

X being not the $\mathfrak{B}$-space. Let $F_\alpha=[\alpha, \omega_1)$ for each $\alpha \in X$, then $\{F_\alpha \mid \alpha \in X\}$ is the monotone decreasing closed collection with vacuous intersection. Suppose that X has the $\mathfrak{B}$-property, then there exists a monotone decreasing open collection $\{G_\alpha \mid \alpha \in X\}$ of X such that $\bigcap_\alpha \overline{G_\alpha}=\emptyset$ and $G_\alpha \supset F_\alpha$ for each $\alpha \in X$. For each $\alpha \in X$, let $f(\alpha)=$minimum of $\{\beta \mid (\beta, \omega_1) \subset G_\alpha\}$, then it is easily seen that f is well defined and the mapping from X to X such that:

(1) $f(\alpha)<\alpha$ for each $\alpha \geqslant 1$,

and

(2) $f(\alpha) \geqslant f(\beta)$ for $\alpha, \beta<\omega_1$ with $\alpha>\beta$.

Since, from the definition of $f(\alpha)$, $(f(\alpha), \omega_1) \subset G_\alpha$ for each α, we have $[f(\alpha)+1, \omega_1) \subset G$, and hence $\bigcap_{\alpha<\omega_1} [f(\alpha)+1, \omega_1) \subset \bigcap_{\alpha<\omega_1} G_\alpha=\emptyset$. This fact means that

(3) $\{f(\alpha) \mid \alpha<\omega_1\}$ is cofinal[5] in $[0, \omega_1)$.

From (1), there exists some element α_0 of X such that

5) The subset A of $[0, \omega_1)$ is said to be *cofinal* in $[0, \omega_1)$ if, for each $\alpha<\omega_1$, there exists an element β of A with $\beta \geq \alpha$.

(4) $\{\alpha \mid \alpha<\omega_1,\ f(\alpha)\leqslant\alpha_0\}$ is cofinal in X.
For α_0, there exists an element α_1 of X such that $f(\alpha_1)>\alpha_0$ (by (3)) and hence, there exists an element α_2 of X such that $\alpha_2>\alpha_1$ and $f(\alpha_2)\leq\alpha_0$ (by (4)). Furthermore, by (2), we have $\alpha_0<f(\alpha_1)\leq f(\alpha_2)\leq\alpha_0$. This is the contradiction.

X being the weak $\mathfrak{B}$*-space.* Let $\mathfrak{F}=\{F_\lambda \mid \lambda\in\Lambda\}$ be an arbitrary monotone decreasing non-empty closed sets of X with the vacuous intersection. Let $f(\alpha)$ be the minimum of $\{\lambda \mid \lambda\in\Lambda,\ [0,\alpha]\cap F_\lambda=\emptyset\}$ for each $\alpha\in X$, then $f(\alpha)$ being well defined is easily seen. We will show that $\Lambda_0=\{f(\alpha)\mid\alpha\in X\}$ is cofinal in Λ. For this purpose, we assume that Λ_0 is not cofinal in Λ, i.e., there exists λ_0 of Λ such that $f(\alpha)\leq\lambda_0$ for each $\alpha\in[0,\omega_1)$, and hence, $[0,\alpha]\cap F_{\lambda_0}=\emptyset$ for each $\alpha\in[0,\omega_1)$, contradicting the non-empty set of F_{λ_0}.

Let α_λ be any fixed element of $f^{-1}(\lambda)$(for each $\lambda\in\Lambda_0$), then $\{\alpha_\lambda\mid\lambda\in\Lambda_0\}$ being cofinal in $[0,\omega_1)$ is clear. Lastly we let

$$G_\lambda=(\alpha_\lambda,\omega_1) \qquad \text{if } \lambda\in\Lambda_0,$$

and

$$G_\lambda=[0,\omega_1) \qquad \text{if } \lambda\in\Lambda-\Lambda_0,$$

then $\{G_\lambda\mid\lambda\}$ is the open subsets of X such that $\bigcap_\lambda G_\lambda=\emptyset$ and $G_\lambda\supset F_\lambda$ for each $\lambda\in\Lambda$. On the other hand, since X is normal, this follows are trivial.

Remark. The space of Example 3 is clearly the α_2-space but not the α_1-space.

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