44. Quasi-normal Analytic Spaces

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1. Introduction. Certain analytic spaces are introduced in [3] (p, 287) for which the weak version of the Riemann extension theorem holds. An example is also given which is not normal. Such spaces are called maximal spaces in [1] (p. 44, p. 178) and some theorems are stated which are extensions from the case of normal spaces. We shall call these spaces quasi-normal. We shall be concerned with three kinds of sheaves on an analytic space; that is, the sheaf of germs of holomorphic functions, the sheaf of germs of continuous and weakly holomorphic functions and the sheaf of germs of weakly holomorphic functions. They are denoted by $\mathcal{O}$, $\mathcal{O}'$ and $\tilde{\mathcal{O}}$, respectively.

We examine the relation between $\mathcal{O}'$ and $\tilde{\mathcal{O}}$ in § 2. In § 3, we first define the quasi-normality at a point by means of $\mathcal{O}_p$ and $\mathcal{O}'_p$, and then the quasi-normal space is defined. Some theorems are stated for which the quasi-normal space is a proper place. In § 4, examples are discussed.

The terminology is that of [4].

2. $\mathcal{O}'_p$ and $\tilde{\mathcal{O}}_p$: Local irreducibility. Let X be an analytic space with the structure sheaf ${}_X\mathcal{O}$. This is also denoted simply by $\mathcal{O}$. We denote by $\mathcal{O}_p$ the stalk of $\mathcal{O}$ at a point $p \in X$. X decomposes into the manifold $\mathfrak{R}(X)$ consisting of regular points and the singular locus $\mathfrak{S}(X)$.

A weakly holomorphic function f on an open subset U of X is a complex-valued function defined and holomorphic in $\mathfrak{R}(X) \cap U$ and locally bounded in U. The sheaf of germs of such functions on X is denoted by ${}_X\tilde{\mathcal{O}}$ or by $\tilde{\mathcal{O}}$. The sheaf of germs of continuous and weakly holomorphic functions on X is denoted by ${}_X\mathcal{O}'$ or $\mathcal{O}'$. Clearly, $\mathcal{O} \subset \mathcal{O}' \subset \tilde{\mathcal{O}}$. X is normal at p if and only if $\mathcal{O}_p = \tilde{\mathcal{O}}_p$. X is a normal space if it is normal at every point of X. It is known that if X is normal at p then X is irreducible there. We prove the following. X is said to be locally irreducible at p if there exists a neighborhood of p at every point of which X is irreducible.

Theorem 1. *Let X be an analytic space and $p \in X$. Then we have*

(1) *X is locally irreducible at p if and only if $\mathcal{O}'_p = \tilde{\mathcal{O}}_p$.*

(2) *The set of points at which X is not locally irreducible is an analytic subvariety of X.*

Proof. It is well known that if X is locally irreducible at p then

$\mathcal{O}'_p = \tilde{\mathcal{O}}_p$ ([5], p. 114). To prove first that $\mathcal{O}'_p = \tilde{\mathcal{O}}_p$ implies the irreducibility of X at p, let X be reducible at p. The germ $\boldsymbol{X}_p$ determined by X at p admits the decomposition into the irreducible branches: $\boldsymbol{X}_p = \boldsymbol{X}_1 \cup \boldsymbol{X}_2 \cup \cdots \cup \boldsymbol{X}_t$, $t \geqq 2$. Further, the decomposition of $\mathcal{R}(\boldsymbol{X}_p)$ into the connected components is given by $\mathcal{R}(\boldsymbol{X}_p) = \boldsymbol{S}_1 \cup \boldsymbol{S}_2 \cup \cdots \cup \boldsymbol{S}_t$ where each $\boldsymbol{S}_i$ is dense in the corresponding branch $\boldsymbol{X}_i$ ([4], p. 116). Let $\boldsymbol{f}$ be a germ of a function at p which takes on the value 1 on $\boldsymbol{S}_1$ and the value 0 on $\mathcal{R}(\boldsymbol{X}_p) - \boldsymbol{S}_1$. Clearly, $\boldsymbol{f} \in \tilde{\mathcal{O}}_p$. But $\boldsymbol{f} \notin \mathcal{O}'_p$, because $\boldsymbol{f}$ does not extend continuously at p.

Let now $(\tilde{X}, \mu)$ be the normalization of X. μ is a proper, holomorphic mapping of $\tilde{X}$ onto X. As is in [1], the quotient space $X' = \tilde{X}/R$ is defined where R is the equivalence relation: $x \sim y$ if and only if $\mu(x) = \mu(y)$. The sheaf ${}_{\tilde{X}}\mathcal{O}/R$ defined on X' is naturally isomorphic with a sheaf on X, which is a coherent analytic sheaf by a result of [2]. Moreover, this sheaf is nothing but the sheaf ${}_X\mathcal{O}'$. We have thus coherent sheaves on X: $\mathcal{O}$, $\mathcal{O}'$ and $\tilde{\mathcal{O}}$.

The set $\{p \in X \mid \mathcal{O}'_p \neq \tilde{\mathcal{O}}_p\}$ is an analytic subvariety of X and therefore the set $\{p \in X \mid \mathcal{O}'_p = \tilde{\mathcal{O}}_p\}$ is open. This implies that X is locally irreducible at p if $\mathcal{O}'_p = \tilde{\mathcal{O}}_p$. Thus, (1) and (2) are proved.

Remark. The locally irreducible space is characterized by $\mathcal{O}' = \tilde{\mathcal{O}}$. If X is 1-dimensional, X is locally irreducible at p if and only if it is irreducible there ([6]). Hence, the irreducibility is also equivalent with $\mathcal{O}'_p = \tilde{\mathcal{O}}_p$ in that case.

3. $\mathcal{O}_p$ and $\mathcal{O}'_p$: Quasi-normality. Definition. An analytic space X is quasi-normal at $p \in X$ if and only if $\mathcal{O}_p = \mathcal{O}'_p$. If X is quasi-normal at every point of X then we shall say that X is a quasi-normal space.

From the coherence of $\mathcal{O}$ and $\mathcal{O}'$ results the following. This and Theorem 1, (2) imply that the set of points at which X is not normal is the union of two analytic subvarieties of X.

Corollary. *The set of points at which X is not quasi-normal is an analytic subvariety of X.*

We state some theorems for which the quasi-normality plays an essential role. Theorem 3 and Theorem 4 are quoted from [1].

Theorem 2. *Let X be a quasi-normal space and A an analytic subvariety of X with $\dim_x A \leqq \dim_x X - 1$ for every $x \in A$. If φ is a continuous mapping of X into an analytic space Y which is holomorphic on $X - A$, then φ is holomorphic on X.*

Theorem 3. *Let X be a quasi-normal space and φ a continuous mapping of X into an analytic space Y. Let μ be a proper, holomorphic mapping of an analytic space Z onto X such that $\varphi \circ \mu$ is holomorphic. Then φ is holomorphic.*

Theorem 4. *Let X be a quasi-normal space and φ a continuous* *mapping of X into an analytic space Y. Let G be the graph of φ, $G=\{(x, \varphi(x)) \mid x \in X\} \subset X \times Y$. If G is an analytic subvariety of $X \times Y$, then φ is holomorphic.*

The theorem of Radó is stated as follows.

Theorem 5. *Let X be a quasi-normal space and φ a continuous mapping of X into an analytic space Y. If φ is holomorphic on $X-\varphi^{-1}(q)$ for some $q \in Y$, then φ is holomorphic on X.*

Proof. Let ${}_X\mathcal{O}$ and ${}_Y\mathcal{O}$ be the structure sheaves of X and Y. φ^* is induced by φ: $\varphi^*(\boldsymbol{h})=\boldsymbol{h} \circ \varphi$ for $\boldsymbol{h} \in {}_Y\mathcal{O}_{\varphi(x)}$, $x \in X$.

We must show that $\varphi^*({}_Y\mathcal{O}_{\varphi(x)}) \subset {}_X\mathcal{O}_x$ for every $x \in \varphi^{-1}(q)$. Let V be a neighborhood of q and let h be a holomorphic function on V. Let $\alpha=h(q)$. For an open neighborhood U of x such that $\varphi(U) \subset V$, $h \circ \varphi$ is continuous on U and holomorphic in $U-\varphi^{-1}(q)$. Since

$$\mathcal{R}(X) \cap U-\varphi^{-1}(q) \supset \mathcal{R}(X) \cap U-(h \circ \varphi)^{-1}(\alpha),$$

$h \circ \varphi$ is holomorphic on $\mathcal{R}(X) \cap U$ by virtue of the theorem of Radó for manifolds. U is quasi-normal and, therefore, $h \circ \varphi$ is holomorphic on U. This completes the proof.

Remark. Above theorems do not hold if the quasi-normality condition is deleted from X. In fact, let $X=\{(z, w) \in \boldsymbol{C}^2 \mid z^3-w^2=0\}$, then X is a locally irreducible subvariety of $\boldsymbol{C}^2$. But X is not quasi-normal at the origin $(0,0)$. The function defined by

$$\varphi(z, w)= \begin{cases} w/z, & (z, w) \in X-(0,0) \\ 0, & z=w=0 \end{cases}$$

is continuous and weakly holomorphic on X, but not holomorphic at $(0,0)$. The graph of the mapping φ of X into $\boldsymbol{C}$ is clearly an analytic subvariety of $X \times \boldsymbol{C}$. Thus, Theorem 4 is not true for this space. Similarly, Theorem 5 does not hold for X, because $\mathcal{R}(X)=X-\varphi^{-1}(0)$.

4. Examples. Let $X_1, \cdots, X_t$ be the irreducible branches of the germ X_p of X. Let f be a germ of a function at p. We denote by f_i the restriction of f on $X_i: f_i=f \mid X_i$, and by ${}_i\mathcal{O}_p$ the ring of germs of holomorphic functions on X_i. ${}_i\mathcal{O}'_p$ has the same meaning.

Lemma. *Suppose that X has the following property:*

(1) X_i are quasi-normal at p, $i=1,2, \cdots, t$.

(2) If f is an arbitrary germ of a continuous function at p such that $f_i \in {}_i\mathcal{O}_p$, $i=1,2, \cdots, t$, then $f \in {}_X\mathcal{O}_p$.

Then, X is quasi-normal at p.

Proof. Let $f \in \mathcal{O}'_p$. Then $f_i \in {}_i\mathcal{O}'_p$ by a theorem of Remmert ([5], p. 127), and therefore $f_i \in {}_i\mathcal{O}_p$ by (1). Hence, $f \in \mathcal{O}_p$ by (2). This completes the proof.

Theorem 6. *Let V be an analytic subvariety of an open subset D of $\boldsymbol{C}^n$ and $p \in V$. Suppose that V_p admits the following irreducible decomposition: $V_p=V_1 \cup \cdots \cup V_t$, where there exist an open polydisk Δ*

$=\Delta(p; r)$ *in* D, *local coordinates* $w_1, \cdots, w_n$ *in* Δ *and subsets* σ_i *of the set* $\{1, 2, \cdots, n\}$ *such that*

$$V_i \cap \Delta = \{z \in \Delta \mid w_j(z)=0, j \in \sigma_i\}, \qquad 1 \leqq i \leqq t.$$

Then, $V \cap \Delta$ *is a quasi-normal space.*

Proof. It is sufficient to prove the theorem for the case in which V is a subvariety of $\Delta=\Delta(0; r)$ in $\boldsymbol{C}^n$; $V=\Delta_1 \cup \cdots \cup \Delta_t$, where $\sigma_i \subset \{1, \cdots, n\}$ and $\Delta_i = \{(z_1, \cdots, z_n) \in \Delta \mid z_j=0, j \in \sigma_i\}$, $1 \leqq i \leqq t$. Moreover, we have only to prove that V is quasi-normal at the origin 0; to do so, it is sufficient to show that if f is a continuous function on V such that the restrictions $f_i = f \mid \Delta_i$, $i=1, \cdots, t$, are holomorphic, then f is holomorphic on V.

For $t=1$, this is clear. We suppose that the assertion is valid for $t-1$ and we consider the case in which

$$V=\Delta_1 \cup \cdots \cup \Delta_{t-1} \cup \Delta_t, \qquad t \geqq 2.$$

Let $V'=\Delta_1 \cup \cdots \cup \Delta_{t-1}$; $f'=f \mid V'$. Since f' is holomorphic on V' from the induction hypothesis and Δ is a Stein manifold, f' extends to a holomorphic function $\tilde{f}'$ on Δ. Let π_t denote the projection of Δ onto Δ_t. We define a holomorphic function F on Δ by

$$F=\tilde{f}'+f_t \circ \pi_t - \tilde{f}' \circ \pi_t,$$

which is seen to be actually an extension of f. In fact, let $x \in \Delta_t$. Then we obtain $\tilde{f}'(x)=\tilde{f}'(\pi_t(x))$ which implies that

$$F(x)=f_t(\pi_t(x))=f(x).$$

Next, let $x \in V'$. Since V' is a union of polydisks (of different dimensions), we have $\pi_t(V') \subset V' \cap \Delta_t$. From this follows that

$$\begin{aligned} \tilde{f}' \circ \pi_t(x) &= f(\pi_t(x)) \\ &= f_t(\pi_t(x)), \end{aligned}$$

hence we have $F(x)=f(x)$. This completes the proof.

Remark. Let $f_1, f_2, \cdots, f_m$ $(m \leqq n)$ be holomorphic functions in a neighborhood U of $0 \in \boldsymbol{C}^n$ such that $f_i(0)=0$, $i=1, 2, \cdots, m$, and $(f_1, f_2, \cdots, f_m)$ is nonsingular. Let

$$V=\{z \in U \mid f_{j_1}(z) f_{j_2}(z) \cdots f_{j_t}(z)=0,\ j_i \in \sigma_i,\ i=1, \cdots, t\},$$

where $\sigma_1, \cdots, \sigma_t$ are subsets of $\{1, 2, \cdots, n\}$. Then, there exist an open polydisk Δ in U and local coordinates $w_1, \cdots, w_n$ in which $w_j=f_j$, $1 \leqq j \leqq m$. V is decomposed into the irreducible branches: $V \cap \Delta = (V_1 \cap \Delta) \cup \cdots \cup (V_t \cap \Delta)$, where

$$V_i \cap \Delta = \{z \in \Delta \mid w_j(z)=0,\ j \in \sigma_i\}, \qquad 1 \leqq i \leqq t.$$

Therefore, $V \cap \Delta$ is a quasi-normal space.

For the above space, $\mathcal{O}_p = \mathcal{O}'_p \subsetneqq \tilde{\mathcal{O}}_p$ at $p=0$. We show that there exist varieties which are reducible and not quasi-normal at p, that is, $\mathcal{O}_p \subsetneqq \mathcal{O}'_p \subsetneqq \tilde{\mathcal{O}}_p$ for some point p.

Lemma. *Let* V *be an analytic subvariety of an open subset* D *of* $\boldsymbol{C}^n$ *and* $p \in V$. *Let* $V=V_1 \cup \cdots \cup V_t$, $t \geqq 2$, *be the irreducible decomposi-*

tion at p for which $V_i \cap V_j = \{p\}$ *if* $i \neq j$. *If V is quasi-normal at p, then* V_i, $1 \leqq i \leqq t$, *are also quasi-normal at p.*

Proof. Let $f_1 \in {}_1\mathcal{O}'_p$. We define a germ f of a continuous function at p by

$$f(x) = \begin{cases} f_1(x), & x \in V_1 \\ f_1(p), & x \in V_i, \qquad i \geqq 2. \end{cases}$$

Since $\mathcal{R}(V) = \bigcup_{i=1}^{t} \mathcal{R}(V_i) - \{p\}$ and V is quasi-normal at p, we have $f \in {}_V\mathcal{O}_p$. Therefore, $f_1 \in {}_1\mathcal{O}_p$, which completes the proof.

Let V_1 be an analytic subvariety of an open subset D of C^n which is not quasi-normal at $p \in V_1$. We can choose a polydisk Δ and coordinates $z_1, \cdots, z_n$ so that

$$V_1 \cap \Delta \cap \{z_1 = \cdots = z_k = 0\} = \{p\}.$$

Let V_2 be any analytic subvariety of $\{z_1 = \cdots = z_k = 0\}$ through the point p and let $V = V_1 \cup V_2$. Then, V is reducible and not quasi-normal at p. An example is given by a subvariety of C^2 defined by the equation: $z(z^3 - w^2) = 0$.

References

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