37. *A New Theorem on Definability in a Positive Second Order Logic with Countable Conjunctions and Disjunctions*

By Nobuyoshi MOTOHASHI

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Introduction. This paper is a sequel to our papers [6]–[8]. So, we shall assume some results and notations stated in these papers.

Let $\mathfrak{L}$ be a fixed second order logic with countable conjunctions and disjunctions, P be a k-ary predicate constant not in $\mathfrak{L}$.

Let $\mathfrak{L}_1$ be the second order logic obtained from $\mathfrak{L}$ by adding P. We assume $\mathfrak{L}$ has only countably many predicate constants and let $X=V_0$, $Y=V_1$ (see [8]).

By Δ, we shall denote the least set Δ of formulas in $\mathfrak{L}_1$ satisfying

1) Every atomic formula *in* $\mathfrak{L}$ and its negation whose free variables are *among* X belong to Δ.

2) Every atomic formula *in* $\mathfrak{L}_1$ and its negation whose free variables are *among* Y belong Δ.

3) Δ is closed under countable conjunctions, countable disjunctions and first order quantifications.

For any $\mathfrak{L}_1$-structure $\mathfrak{A}$, let $\mathfrak{A}\lceil\mathfrak{L}$ be the reduct of $\mathfrak{A}$ to $\mathfrak{L}$. Let T be a countable set of *negative* sentences in $\mathfrak{L}_1$. Then our main theorem can be expressed as follows;

Main theorem. *The following two conditions are equivalent*;

($*$) *For any models* $\mathfrak{A}$, $\mathfrak{B}$ *of* T, $\mathfrak{A}\lceil\mathfrak{L}=\mathfrak{B}\lceil\mathfrak{L}$ *and* $\mathfrak{A}\cong\mathfrak{B}$. *imply* $\mathfrak{A}=\mathfrak{B}$.

($**$) $T\vdash_{\mathfrak{L}_1}(\forall u_1)\cdots(\forall u_k)(P(u_1,\cdots,u_k)\equiv\theta(u_1,\cdots,u_k))$ *for some* $\theta(x_1,\cdots,x_k)\in\Delta$, *where* $V(\theta)\subseteqq\{x_1,\cdots,x_k\}\subseteqq X$.

Especially if T *is a set of finitary sentences, we can take* θ *above as a finitary sentence in* Δ.

First of all we should remark that the condition ($**$) is not an explicit definition of P in $\mathfrak{L}$ because θ may have the predicate P. But we can not take $P(x_1,\cdots,x_k)$ as θ in ($**$) because $P(x_1,\cdots,x_k)\notin\Delta$. On the contrary, $P(y_1,\cdots,y_k)\in\Delta$ for any $y_1,\cdots,y_k$ in Y.

Our main theorem can be considered as an extension of Svenonius' definability theorem (cf. Kochen [2], Motohashi [5]) to $L_{\omega_1\omega}$ because we can prove Svenonius' theorem from our main theorem just as we can prove Beth's definability theorem from Craig's interpolation theorem.

Finally we shall show some types of extensions of Svenonius' theorem and Chang-Makkai's theorem do not hold in $L_{\omega_1\omega}$ (cf. Chang [1], Kueker [3] and Makkai [4]).

§1. Some proofs. (I) A proof of main theorem. We shall use $\mathfrak{L}_1^I$ in [8]. Let

$$\begin{aligned}\Psi=\{&(\forall u)(\exists v)I_0(u,v),\ (\forall u)(\exists v)I_1(u,v),\\ &(\forall v)(\exists u)I_0(u,v),\ (\forall v)(\exists u)I_1(u,v),\\ &(\forall\vec{u})(\mathsf{A}\vec{v})(I_0(\vec{u},\vec{v})\wedge\varphi^1(\vec{u})\supset\varphi^2(\vec{v})),\\ &(\forall\vec{u})(\forall\vec{v})(I_1(\vec{u},\vec{v})\wedge\psi^1(\vec{u})\supset\psi^2(\vec{v})),\end{aligned}$$

where φ is an atomic formula or its negation *in* $\mathfrak{L}$ and ψ is an atomic formula or its negation *in* $\mathfrak{L}_1\}$.

Then Ψ is a first order primitive set and $\Delta(\Psi)=\Delta$.
Let $\vec{x}$, $\vec{y}$ be two sequences of distinct free variables of the same length k.

By using Ψ

$$\begin{aligned}(*)&\Longleftrightarrow\Psi\vdash_{\mathfrak{L}_1^I}(\wedge T)^1,\ (\wedge T)^2\to(\forall\vec{u})(\forall\vec{v})(P^1(\vec{u})\wedge I_0(\vec{u},\vec{v})\supset P^2(\vec{v})).\\ &\Longleftrightarrow\Psi\vdash_{\mathfrak{L}_1^I}(\wedge T)^1,\ P^1(\vec{x}),\ I_0(\vec{x},\vec{y})\to((\wedge T)^2\supset P^2(\vec{y})).\\ &\Longleftrightarrow\ \vdash_{\mathfrak{L}_1}T,\ P(\vec{x})\to\theta(\vec{x})\text{ and }\vdash_{\mathfrak{L}_1}\theta(\vec{y})\to(\wedge T\supset P(\vec{y}))\text{ for some }\theta\in\Delta.\\ &\Longleftrightarrow T\vdash_{\mathfrak{L}_1}P(\vec{x})\supset\theta(\vec{x})\text{ and }T\vdash_{\mathfrak{L}_1}\theta(\vec{y})\supset P(\vec{y})\text{ for some }\theta\in\Delta.\\ &\Longleftrightarrow T\vdash_{\mathfrak{L}_1}(\forall\vec{u})(P(\vec{u})\equiv\theta(\vec{u}))\text{ for some }\theta\in\Delta.\\ &\Longleftrightarrow(**)\end{aligned}$$

(Especially if T is finitary, obviously we can take θ as a finitary formula.)

(II) A proof of the fact that our main theorem implies Svenonius' theorem. At first we should remark that every finitary sentence in Δ can be expressed by the following type formulas:

$$(\theta_{11}\vee\theta_{21})\wedge\cdots\wedge(\theta_{1n}\vee\theta_{2n})\qquad\text{where }V(\theta_{1i})\subseteq X,\ V(\theta_{2i})\subseteq Y,\ i=1,2,\cdots,n.$$

Assume that T is a set of finitary sentences and $\vec{x}$ is a sequence of distinct free variables of the length k.

Then

$$\begin{aligned}(*)&\Longleftrightarrow T\vdash_{\mathfrak{L}_1}P(\vec{x})\supset\theta(\vec{x})\quad\text{and}\quad T\vdash_{\mathfrak{L}_1}\theta(\vec{x})\supset P(\vec{x}).\\ &\Longleftrightarrow T\vdash_{\mathfrak{L}_1}P(\vec{x})\supset(\theta_{11}(\vec{x})\vee\theta_{21})\wedge\cdots\wedge(\theta_{1n}(\vec{x})\vee\theta_{2n})\end{aligned}$$

and

$$\begin{aligned}&T\vdash_{\mathfrak{L}_1}(\theta_{11}(\vec{x})\vee\theta_{21}\wedge\cdots\wedge(\theta_{1n}(\vec{x})\vee\theta_{2n})\supset P(\vec{x}).\\ \Longleftrightarrow\ &T\vdash_{\mathfrak{L}_1}P(\vec{x})\wedge\neg\theta_{2i}\cdot\supset\theta_{1i}(\vec{x}),\qquad i=1,2,\cdots,n\end{aligned}$$

and

$$T\vdash_{\mathfrak{L}_1}\bigwedge_{i=1}^{n}(\theta_{1i}(\vec{x})\vee\theta_{2i})\supset P(\vec{x}).$$

For each $I\subseteq\{1,2,\cdots,n\}$ let

$$\theta_{1I}(\vec{x})=\bigwedge_{i\in I}\theta_{1i}(\vec{x}),\ \theta_{2I}=\bigwedge_{i\in I}\neg\theta_{2i}\wedge\bigwedge_{i\notin I}\theta_{2i}.$$

Assume $(*)$. Then by above, we have

$$T\vdash_{\mathfrak{L}_1} P(\vec{x})_\wedge \neg\theta_{2i}. \supset \theta_{1i}(\vec{x}), \qquad i=1,2,\cdots,n$$

and

$$T\vdash_{\mathfrak{L}_1} \bigwedge_{i=1}^{n}(\theta_{1i}(\vec{x})^\vee\theta_{2i}) \supset P(\vec{x}).$$

Hence

$$\left.\begin{array}{l} T\vdash_{\mathfrak{L}_1} P(\vec{x})_\wedge\theta_{2I}. \supset\theta_{1I}(\vec{x}) \\ \text{and} \\ T\vdash_{\mathfrak{L}_1}\theta_{2I\wedge}\theta_{1I}(\vec{x}). \supset P(\vec{x}) \end{array}\right\} \text{ for any } I\subseteq\{1,2,\cdots,n\}.$$

Therefore

$$T\vdash_{\mathfrak{L}_1}\theta_{2I}\supset(\forall\vec{u})(P(\vec{u})\equiv\theta_{1I}(\vec{u})) \qquad \text{for any } I.$$

We get

$$T\vdash_{\mathfrak{L}_1}\bigvee_I\theta_{2I}\supset\bigvee_I(\forall\vec{u})(P(\vec{u})\equiv\theta_{1I}(\vec{u})).$$

But

$$\vdash_{\mathfrak{L}_1}\bigvee_I\theta_{2I}.$$

Hence

$$T\vdash_{\mathfrak{L}_1}\bigvee_I(\forall\vec{u})(P(\vec{u})\equiv\theta_{1I}(\vec{u})).$$

So, we get

$$(*)\Rightarrow T\vdash_{\mathfrak{L}_1}\bigvee_{j=1}^{m}(\forall\vec{u})(P(\vec{u})\equiv\theta_j(\vec{u})) \qquad \text{for some } \theta_j(\vec{x}) \text{ in } \mathfrak{L},\ j=1,\cdots,m.$$

Obviously the right statement implies the left. So,

$$(*)\Longleftrightarrow T\vdash_{\mathfrak{L}_1}\bigvee_{j=1}^{m}(\forall\vec{u})(P(\vec{u})\equiv\theta_j(\vec{u})) \quad \text{for some } \theta_j(\vec{x}),\ j=1,\cdots,m \text{ in } \mathfrak{L}.$$

This is Svenonius' definability theorem extended by Kochen [2]. (Notice that T may have second order quantifiers.)

(III) Svenonius' type theorem and Chang-Makkai's type theorem do not hold in $L_{\omega_1\omega}$. For simplicity assume $k=1$ and the set of predicate constants in $\mathfrak{L}$ are $\{P_n\}_{n<\omega}$ where P_n are unary for each $n<\omega$. For any $\mathfrak{L}$-structure $\mathfrak{A}$ and any $S\subseteq|\mathfrak{A}|$, let $(\mathfrak{A},S)_T=$the power of the sets $S_1\subseteq|\mathfrak{A}|$ such that

$$(\mathfrak{A},S)\cong(\mathfrak{A},S_1).$$

Now we shall consider the following two statements;

$$(***) \qquad T\vdash_{\mathfrak{L}_1}\bigvee_{k<\omega}(\forall\vec{u})(P(\vec{u})=\theta_k(\vec{u})) \quad \text{for some } \theta_k(\vec{x}),\ k<\omega \text{ in } \mathfrak{L}.$$

$$(****) \qquad T\vdash_{\mathfrak{L}_1}(\exists\vec{v})\bigvee_{k<\omega}(\forall\vec{u})(P(\vec{u})\equiv\theta_k(\vec{u},\vec{v})) \quad \text{for some } \theta_k(\vec{x},\vec{y}),\ k<\omega \text{ in } \mathfrak{L}.$$

Then we can consider

"$(***)$ is equivalent to $(*)$" as a generalization of Svenonius' theorem to $L_{\omega_1\omega}$ and

"$(****)$ is equivalent to $(\mathfrak{A},S)_T\leqq|\mathfrak{A}|$ for any model $(\mathfrak{A},S)$ of T", as a generalization of Chang-Makkai's theorem to $L_{\omega_1\omega}$.

We shall prove in the following that these two generalizations do not hold in $L_{\omega_1\omega}$.

Let $T=\{(\forall u)(P(u)\equiv \bigwedge_{n<\omega}(\neg P_n(u)\vee(\exists v)(P(v)\wedge P_n(v)))\}$.
Then obviously T satisfies $(**)$, hence $(*)$ by our main theorem.
Hence $(\mathfrak{A}, S)_T=1\leqq|\mathfrak{A}|$ for any model $(\mathfrak{A}, S)$ of T.
We want to show that $(***)$ and $(****)$ don't hold for this T.
For each $I\subseteqq\omega$, let $\mathfrak{A}_I$ be $|\mathfrak{A}_I|=\omega$, $\mathfrak{A}_I(P_n)=\{n\}$, $\mathfrak{A}_I(P)=I$, $(n<\omega)$.
Then obviously $\mathfrak{A}$ is a model of T, $\mathfrak{A}_I\lceil\mathfrak{L}=\mathfrak{A}_J\lceil\mathfrak{L}$ and $\mathfrak{A}_I\neq\mathfrak{A}_J$ for any $I\neq J\subseteqq\omega$.
Let $\mathfrak{A}_I\lceil\mathfrak{L}=\mathfrak{A}$. Then the class of all sets definable by $\{\theta_k(x)\}_{k<\omega}$ or $\{\theta_k(x,\vec{y})\}_{k<\omega}$ in $\mathfrak{A}$ is at most countable.
On the other hand, $\{I\,;\,I\subseteqq\omega\}$ is uncountable.
These mean that $(***)$ and $(****)$ don't hold.

References

[1] C. C. Chang: Some new results in the theory of definition. Bull. Amer. Math. Soc., **70**, 808–813 (1964).

[2] S. Kochen: Topics in the theory of definition. Proc. of Model Theory Symposium, Berkeley, **1963**, 170–176 (1965).

[3] D. Kueker: Definability, automorphisms and infinitary languages, in: J. Barwise (editor). Syntax and Semantics of Infinitary languages, Springer Lectures Notes, **72**, 152–165 (1968).

[4] M. Makkai: A generalization of a theorem of Beth. Acta Math. Acad. Sci., Hungar., **15**, 227–236 (1964).

[5] N. Motohashi: A theorem in the theory of definition. J. Math. Soc. Japan, **22**, 490–494 (1970).

[6] ——: Object logic and Morphism logic (to appear).

[7] ——: Interpolation theorem and characterization theorem (to appear).

[8] ——: Model theory in a positive second order logic with countable conjunctions and disjunctions (to appear).