36. *On Random Ergodic Theorems for a Random Quasi-semigroup of Linear Contractions*

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(Comm. by Kôsaku YOSIDA, M. J. A., March 13, 1972)

1. The purpose of the present paper is to state a random ergodic theorem and a random local ergodic theorem for a random quasi-semigroup of linear contractions associated with a semiflow of measure preserving transformations.

2. We consider a measure space $(R^+, \mathcal{M}, dt)$ where $R^+=[0, \infty)$, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable subsets of R^+ and dt the Lebesgue measure on $\mathcal{M}$. We consider also two σ-finite measure spaces $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \mu)$.

Let $\{\varphi_t : t \in R^+\}$ be a semiflow of measure preserving transformations defined in such a way that

(φ.1) for every t, φ_t is a measure preserving transformation in X and φ_0 is the identity;

(φ.2) for every s, t, $\varphi_{s+t}=\varphi_s\varphi_t$;

(φ.3) $\varphi_t x$ is a measurable mapping from $R^+\otimes X$ into X.

Let $\{T(t, x) : (t, x) \in R^+\otimes X\}$ be a random quasi-semigroup of linear contractions on $L^1(Y)$ associated with $\{\varphi_t : t \in R^+\}$ defined in such a way that

(T.1) for every t and λ-a.a.x, $T(t, x)$ is a linear contraction on $L^1(Y)$ and $T(0, x)$ is the identity;

(T.2) for every s, t and λ-a.a.x, $T(s+t, x)=T(s, x)T(t, \varphi_s x)$;

(T.3) for every fixed t, $T(t, x)$ is strongly $\mathcal{A}$-measurable in X;

(T.4) for λ-a.a. fixed x, $T(t, x)$ is strongly t-continuous in R^+.

Then, given $f \in L^1(X\otimes Y)$ and given t, $f(\varphi_t x, \cdot) \in L^1(Y)$ λ-a.e. and so we can define $T(t, x)f(\varphi_t x, \cdot)$ λ-a.e. Moreover we can choose a function $g(t, x, y)$ on $R^+\otimes X\otimes Y$ satisfying that

(1) $g(t, x, y)$ is $\mathcal{M}\otimes\mathcal{A}\otimes\mathcal{B}$-measurable;

(2) for every t, there exists a subset N_t of X with λ-measure zero such that, for every $x \notin N_t$,

$$T(t, x)f(\varphi_t x, y)=g(t, x, y) \qquad \mu\text{-a.e.}.$$

The existence of such a $g(t, x, y)$ will be shown by Lemmas 1 and 4. $g(t, x, y)$ is called a *good version* of $T(t, x)f(\varphi_t x, y)$ and denoted by $[T(t, x)f(\varphi_t x, y)]$.

Now we consider two properties:

(T.5) for every t and λ-a.a.x, $T(t, x)$ is a positive operator on $L^1(Y)$;
(T.6) for every t and λ-a.a.x, $T(t, x)$ is a contraction on $L^\infty(Y)$ in the sense of that

$$\operatorname*{ess\,sup}_{y \in Y} |T(t, x)f(y)| \leqslant \operatorname*{ess\,sup}_{y \in Y} |f(y)|$$

for all $f \in L^1(Y) \cap L^\infty(Y)$.

Then we have

Theorem 1 (Random ergodic theorem). *Let $\{T(t, x): (t, x) \in R^+ \otimes X\}$ be a random quasi-semigroup of linear contractions on $L^1(Y)$ associated with $\{\varphi_t : t \in R^+\}$. Further, assume* (T.6). *Then, for every $f \in L^1(X \otimes Y)$, there exists a function $f^* \in L^1(X \otimes Y)$ such that*

$$\lim_{s \to +\infty} \frac{1}{s} \int_0^s [T(t, x)f(\varphi_t x, y)]dt = f^*(x, y) \qquad \lambda \otimes \mu\text{-a.e.}.$$

Theorem 2 (Random local ergodic theorem). *Let $\{T(t, x): (t, x) \in R^+ \otimes X\}$ be a random quasi-semigroup of linear contractions on $L^1(Y)$ associated with $\{\varphi_t : t \in R^+\}$. Further, assume* (T.5) *or* (T.6). *Then, for every $f \in L^1(X \otimes Y)$,*

$$\lim_{s \to +0} \frac{1}{s} \int_0^s [T(t, x)f(\varphi_t x, y)]dt = f(x, y) \qquad \lambda \otimes \mu\text{-a.e.}.$$

3. In this section we show the existence of good versions and prove Theorems 1 and 2.

Lemma 1. *Let t be arbitrarily fixed. Then, for every $f \in L^1(X \otimes Y)$, there exist a function $g_t \in L^1(X \otimes Y)$ and a subset M_t of X with λ-measure zero such that, for every $x \notin M_t$,*

$$T(t, x)f(\varphi_t x, y) = g_t(x, y) \qquad \mu\text{-a.e.}.$$

Such a function g_t is uniquely determined except on a set of $\lambda \otimes \mu$-measure zero. Thus a mapping S_t from $L^1(X \otimes Y)$ into itself can be defined by

$$S_t f = g_t.$$

This can be proved on making use of (T.1) and (T.3). Refer to [6, Lemma 3.2].

Lemma 2. *$\{S_t : t \in R^+\}$ is a semigroup of linear contractions on $L^1(X \otimes Y)$. Moreover, if* (T.5) *is assumed, S_t is a positive operator on $L^1(X \otimes Y)$, and if* (T.6) *is assumed, S_t is a contraction on $L^\infty(X \otimes Y)$ in the sense of that*

$$\operatorname*{ess\,sup}_{(x,y) \in X \otimes Y} |(S_t f)(x, y)| \leqslant \operatorname*{ess\,sup}_{(x,y) \in X \otimes Y} |f(x, y)|$$

for all $f \in L^1(X \otimes Y) \cap L^\infty(X \otimes Y)$.

Proof. S_t is clearly linear and further, when (T.5) holds, it is clearly positive. By (T.1) it holds that, for every $f \in L^1(X \otimes Y)$,

$$\|S_t f\|_{L^1(Y)} = \|T(t, x)f(\varphi_t x, y)\|_{L^1(Y)} \leqslant \|f(\varphi_t x, y)\|_{L^1(Y)} \qquad \lambda\text{-a.e.},$$

and so

$$\|S_t f\|_{L^1(X \otimes Y)} \leqslant \|f(\varphi_t x, y)\|_{L^1(X \otimes Y)} = \|f\|_{L^1(X \otimes Y)}.$$

Hence S_t is a contraction on $L^1(X\otimes Y)$. When (T.6) holds, we can show similarly that S_t is also a contraction on $L^\infty(X\otimes Y)$.

Next, we prove the semigroup property of S_t. Let $f\in L^1(X\otimes Y)$. Then, by (T.2) it holds that, in the space $L^1(Y)$,

$$\begin{aligned}(S_tf)(x,\cdot)&=T(t,x)f(\varphi_t x,\cdot) && \lambda\text{-a.e.},\\ (S_sS_tf)(x,\cdot)&=T(s,x)((S_tf)(\varphi_s x,\cdot))\\ &=T(s,x)(T(t,\varphi_s x)f(\varphi_t\varphi_s x,\cdot)) && \lambda\text{-a.e.},\\ (S_{s+t}f)(x,\cdot)&=T(s+t,x)f(\varphi_{s+t}x,\cdot) && \lambda\text{-a.e.},\end{aligned}$$

and so

$$(S_sS_tf)(x,\cdot)=(S_{s+t}f)(x,\cdot) \qquad \lambda\text{-a.e..}$$

Hence, in the space $L^1(X\otimes Y)$,

$$S_sS_tf=S_{s+t}f.$$

Lemma 3. *S_t is strongly t-continuous in R^+.*

Proof. Let $f\in L^1(X\otimes Y)$. Then, by (T.1),

$$\begin{aligned}\|S_sf-S_tf\|_{L^1(X\otimes Y)}&=\|\,\|S_sf-S_tf\|_{L^1(Y)}\|_{L^1(X)}\\ &=\|\,\|T(s,x)f(\varphi_s x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}\\ &\leqslant\|\,\|T(s,x)f(\varphi_s x,y)-T(s,x)f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}\\ &\quad+\|\,\|T(s,x)f(\varphi_t x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}\\ &\leqslant\|\,\|f(\varphi_s x,y)-f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}\\ &\quad+\|\,\|T(s,x)f(\varphi_t x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}\\ &=\|f(\varphi_s x,y)-f(\varphi_t x,y)\|_{L^1(X\otimes Y)}\\ &\quad+\|\,\|T(s,x)f(\varphi_t x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}.\end{aligned}$$

Now, if we define $(V_tf)(x,y)=f(\varphi_t x,y)$ for $t\in R^+$ and $f\in L^1(X\otimes Y)$, we see that $\{V_t\colon t\in R^+\}$ is a semigroup of linear contractions on $L^1(X\otimes Y)$ and that V_t is strongly $\mathscr{M}$-measurable and so strongly t-continuous. Hence

$$\lim_{s\to t}\|f(\varphi_s x,y)-f(\varphi_t x,y)\|_{L^1(X\otimes Y)}=0.$$

On the other hand, by (T.4),

$$\lim_{s\to t}\|T(s,x)f(\varphi_t x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}=0 \qquad \lambda\text{-a.e.},$$

and, by (T.1),

$$\begin{aligned}&\|T(s,x)f(\varphi_t x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}\\ &\qquad\leqslant 2\|f(\varphi_t x,y)\|_{L^1(Y)}\in L^1(X),\end{aligned}$$

because $\|\,\|f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}=\|f\|_{L^1(X\otimes Y)}$. Hence, by Lebesgue convergence theorem,

$$\lim_{s\to t}\|\,\|T(s,x)f(\varphi_t x,y)-T(t,x)f(\varphi_t x,y)\|_{L^1(Y)}\|_{L^1(X)}=0.$$

Therefore

$$\lim_{s\to t}\|S_sf-S_tf\|_{L^1(X\otimes Y)}=0.$$

Lemma 4. *For every $f\in L^1(X\otimes Y)$ there exists a measurable function $g(t,x,y)$ on $R^+\otimes X\otimes Y$ such that, for every t,*

$$(S_tf)(x,y)=g(t,x,y) \qquad \lambda\otimes\mu\text{-a.e..}$$

Such a function $g(t,x,y)$ is uniquely determined except on a set of

$dt\otimes\lambda\otimes\mu$- measure zero.

For the proof, see [1], [4].

By virtue of Lemmas 1 and 4, given t and $f\in L^1(X\otimes Y)$, $T(t,x)f(\varphi_t x, y)$ has its good version $[T(t,x)f(\varphi_t x, y)]=g(t,x,y)$. Thus, in order to obtain Theorems 1 and 2 it suffices to apply Dunford-Schwartz ergodic theorem [2, Theorem 5 in § 4] and Krengel-Ornstein local ergodic theorem [3]–[5], to the present semigroup $\{S_t : t\in R^+\}$ on considering a good version $[T(t,x)f(\varphi_t x, y)]$ for $f\in L^1(X\otimes Y)$.

The results in the present paper are extended to the case of a multi-parameter random quasi-semigroup. In the case, Dunford-Schwartz ergodic theorem [2, Theorem 10 and 17 in § 4] and Terrell local ergodic theorem [5] are used for the proof.

References

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