73. On the Existence of Quasiperiodic Solutions of Nonlinear Hyperbolic Partial Differential Equations

By Masaru YAMAGUCHI

(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1972)

1. Introduction.

In this note we shall consider a global property, that is, the quasiperiodic property, of the solutions of the following quasilinear one dimensional wave equation with dissipative term αu_t, where α is a constant:

(1) $$M(u)=u_{tt}-u_{xx}+\alpha u_t=h(x,t,u,u_x,u_t),$$

where h is quasiperiodic with basic frequencies $\omega_1,\cdots,\omega_m$ in t. We shall show the existence of such quasiperiodic solutions of the form (1) that have the same basic frequencies as h and satisfy the boundary conditions $u(0,t)=u(\pi,t)=0$. These solutions are classical solutions.

The case $m=1$ is the periodic case and was already solved by Rabinowitz [1], [2]. Especially, in [2] equation is strictly nonlinear.

2. Notations and definitions.

Definition. $f(x,t)$ is called *quasiperiodic* with basic frequencies $\omega_1,\cdots,\omega_m$ in t, if there exists a function $F(x,\theta_1,\cdots,\theta_m)$ such that $f(x,t)=F(x,\omega_1 t,\cdots,\omega_m t)$, where $F(x,\theta_1,\cdots,\theta_m)$ is a continuous function of period 2π in $\theta_1,\cdots,\theta_m$. Basic frequencies $\omega_1,\cdots,\omega_m$ are real numbers. We shall denote by $\mathcal{B}^k(\omega_1,\cdots,\omega_m)$ the class of $f(x,t)$ for which $\mathcal{F}(x,\theta_1,\cdots,\theta_m)$ is C^k-class in $x,\theta_1,\cdots\theta_m$ and by $\mathcal{F}^k(\omega_1,\cdots,\omega_m)\subset\mathcal{B}^k(\omega_1,\cdots,\omega_m)$ the class of $f(x,t)$ which is 2π-periodic in $x(1\leqq k\leqq\infty)$. Every $f(x,t)\in\mathcal{F}^k$ is expanded in the Fourier series if $k\geqq 1$:

$$f(x,t)=\sum_{j\in Z,k\in Z^m} f_{jk}e^{ijx}e^{i(\omega,k)t}.$$

We introduce the norms in F^k by $\|f\|=\sum|f_{jk}|$ and

$$\|f\|_1=\|f\|+\|f_x\|+\|f_t\|.$$

Now we assume that $h(x,t,p,q,r)$ is in the form

$$f(x,t)+g(x,t,p,q,r),\ f(x,t)\not\equiv 0.$$

Then we can represent $g(x,t,p,q,r)$ in the form $G(x,\omega_1 t,\cdots,\omega_m t,p,q,r)$, where $G(x,\theta_1,\cdots,\theta_m,p,q,r)$ is continuous and 2π-periodic in $\theta_1,\cdots,\theta_m$. Further we assume that $f(x,t)$ and $g(x,t,u,u_x,u_t)$ vanish at the boundary $x=0,x=\pi$.

3. The existence of quasiperiodic solutions.

3.1. At first we consider the case where the forcing term $h(x,t,u,u_x,u_t)$ does not depend on u,u_x,u_t:

(2) $$M(u)=u_{tt}-u_{xx}+\alpha u_t=f(x,t).$$

As for (2) we have two propositions:

Proposition 1. *If $f(x,t)$ belongs to $\mathcal{F}^\infty(w_1,\cdots,\omega_m)$ and $\alpha\neq 0$, then (2) has a unique classical solution $u=u(x,t)\in\mathcal{F}^\infty(\omega_1,\cdots,\omega_m)$. This solution satisfies the estimate:*

$$\|u\|_1\leqq C(\alpha)\|f\|,\quad i.e.\quad \|Lf\|_1\leqq C(\alpha)\|f\|,\quad where\quad L=M^{-1}$$

and $C(\alpha)$ is a constant depending only on α.

Proposition 2. *If $f(x,t)$ belongs to $\mathcal{F}^\infty(\omega_1,\cdots,\omega_m)$ and $\alpha=0$, then (2) has a unique solution $u=u(x,t)\in\mathcal{F}^\infty(\omega_1,\cdots,\omega_m)$, provided that $1,\omega_1,\cdots,\omega_m$ satisfy the irrationality condition: For some constants $\gamma>0$ and $\tau>m$, $(1,\omega_1,\cdots,\omega_m)$ satisfy $|k_0+k_1\omega_1+\cdots+k_m\omega_m|\geqq\gamma(|k_0|+\cdots+|k_m|)^{-\tau}$ for all $(k_0,\cdots,k_m)\in \boldsymbol{Z}^{m+1}$.*

Above two propositions are proved by comparing the Fourier coefficients and using the estimates of them. Here we need the following lemma:

Lemma. *Let $\mathcal{F}(x_1,\cdots,x_s)$ be C^∞-function of period 2π in $x_1,\cdots,x_s$. Then the Fourier coefficients f_k of $F=\sum_{k\in \boldsymbol{Z}^s}f_k e^{i(k,x)}$ satisfy the estimates:*

$$|f_k|\leqq\frac{\hat{C}(s)\sup\limits_{|\sigma|\leqq Ns}\sup\limits_{x_1\cdots x_s}|D^\sigma F|}{(1+|k_1|)^N\cdots(1+|k_s|)^N}$$

for any natural number N, where $k=(k_1,\cdots,k_s)\in \boldsymbol{Z}^s$ and

$$D^\sigma=\frac{\partial^{|\sigma|}}{\partial x_1^{\sigma_1}\cdots\partial x_s^{\sigma_s}}.$$

Remark. Irrationality condition in Proposition 2 is not unreasonable, since almost all $(\omega_0,\omega_1,\cdots,\omega_m)\in \boldsymbol{R}^{m+1}$ satisfy this condition.

3.2. Now we consider the quasilinear case. We assume the following:

$$(\mathrm{C})\begin{cases} f(x,t)\text{ and }g(x,t,p,q,r)\text{ belong to }F^\infty(\omega_1,\cdots,\omega_m);\\ G(x,\theta_1,\cdots,\theta_m,p,q,r)\text{ is analytic in }p,q,r\text{ in neighborhood}\\ \text{of }(0,0,0)\text{ with its derivatives of sufficiently high orders.}\end{cases}$$

Our results are as follows. Assume $\alpha\neq 0$.

Theorem A. *Suppose that in addition to the condition (C) the convergence radius R of the power series which expresses $G(\cdots,p,q,r)$ satisfies the inequality $2C(\alpha)\|f\|<R$. If $g(x,t,p,q,r)$ is of the form $\varepsilon\tilde{g}(x,t,p,q,r)$ for sufficiently small $\varepsilon>0$, that is, (1) is a perturbed equations of (2), then (1) has a unique solution $u=u(x,t)\in\mathcal{F}^\infty(\omega_1,\cdots,\omega_m)$. This satisfies the estimate: $\|u\|_1\leqq 2C(\alpha)\|f\|$, where $C(\alpha)$ is the same as in Proposition 1.*

Theorem B. *Suppose that in addition to (C), the power series $G(\cdots,p,q,r)$ begins with order $k\geqq 2$. If $\|f\|$ is sufficiently small, then we obtain the same conclusion as that of Theorem A.*

These theorems can be proved by applying the well-known Picard's iteration method and the estimates from Proposition 1, lemma and

Cauchy's estimation formula.

Finally we consider the stability of the above solution $u(x,t)$ of Theorem A. Suppose that there exists a second global solution $v(x,t)$ of (1) which satisfies the boundary conditions. Then we have the following result:

Theorem C. *There exists a constant* $\beta(\alpha,\varepsilon,g)>0$ *such that* $|u(x,t)-v(x,t)|\leqq\gamma e^{-(\beta/2)t}$ *for sufficiently small* $\varepsilon>0$ *and the initial values of* v *sufficiently close that of* u*, where* γ *depends on* α *and* v *initially.*

For the proof, see [1].

References

[1] Rabinowitz, P. H.: Periodic solutions of nonlinear hyperbolic partial differential equations. I. Comm. Pure Appl. Math., **20**, 145–205 (1967).

[2] ——: Periodic solutions of nonlinear hyperbolic partial differential equations. II. Comm. Pure Appl. Math., **22**, 15–39 (1969).

[3] Moser, J.: On the theory of quasiperiodic motions. SIAM Review, **8**(2), 145–172 (1966).