# 80. A Remark on the Asymptotic Behavior of the Solution of $\ddddot{x}+f(\dddot{x})\dddot{x}+\phi(\dot{x},\ddot{x})+g(\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x})$

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**1. Introduction.** This paper is concerned with the equation of the form

(1.1) $$\ddddot{x}+f(\dddot{x})\dddot{x}+\phi(\dot{x},\ddot{x})+g(\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x})$$

where $f, \phi, g, h$ and $p$ are continuous real-valued functions depending only on the arguments shown, and the dots indicate differentiation with respect to the independent variable $t$.

We shall investigate conditions under which all solutions of (1.1) tend to zero as $t\to\infty$. Much work has been done on the asymptotic properties of non-linear differential equations of the fourth order and many of these conditions are summerized in [7, Kapitel 6].

M. Harrow [5] established conditions under which every solution of the equation

(1.2) $$\ddddot{x}+a\dddot{x}+f(\ddot{x})+g(\dot{x})+h(x)=p(t) \qquad (p(t)\equiv 0)$$

tends to zero as $t\to\infty$. A. S. C. Sinha and R. G. Hoft [8] also considered the asymptotic stability of the zero solution of the equation

(1.3) $$\ddddot{x}+f(\dddot{x})\dddot{x}+\phi(\dot{x},\ddot{x})\ddot{x}+\psi(\dot{x})+\theta(x)=p(t) \qquad (p(t)\equiv 0).$$

In [1], M. A. Asmussen studied the behavior as $t\to\infty$ of the solution of the equation

(1.4) $$\ddddot{x}+f(\dddot{x})\dddot{x}+a_2\ddot{x}+g(\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x})$$

where $a_2$ is a positive constant. In this note the same conclusion for the more general equation (1.1) are obtained under the conditions slightly weaker than those of [5], [8] and [1].

**2. Assumptions and Theorem.** Throughout this paper we shall make the following assumptions:

( I ) the function $f(z)$ is continuous in $R^1$,

(II) the functions $\phi(y,z)$ and $\dfrac{\partial\phi}{\partial y}(y,z)$ are continuous in $R^2$,

(III) $g(y)$ is a $C^1$-function in $R^1$,

(IV) $h(x)$ is a $C^1$-function in $R^1$,

( V ) the function $p(t,x,y,z,w)$ is continuous in $[0,\infty)\times R^4$.

Henceforth the following notations are used;

$$g_1(y)=\frac{g(y)}{y} \quad (y\neq 0), \qquad g_1(0)=g'(0),$$

$$f_1(z)=\frac{1}{z}\int_0^z f(\zeta)d\zeta \quad (z\neq 0), \qquad f_1(0)=f(0).$$

**Theorem.** *Assume that the assumptions* (I)~(V) *hold and that there exist positive constants such that*

( i ) $f(z)\geqq a>0 \qquad (z\in R^1)$,

( ii ) $g_1(y)\geqq c>0 \quad (y\in R^1), \quad g(0)=0, g'(y)\geqq 0$,

( iii ) $xh(x)>0, \quad \int_0^x h(\xi)d\xi\to\infty$ *as* $|x|\to\infty$,

$d-\frac{a\delta_0}{2c}<h'(x)\leqq d$,

( iv ) $\phi_y(y,z)\leqq 0, \quad \phi(y,0)=0$ *in* $R^2$,

( v ) $0\leqq\frac{\phi(y,z)}{z}-b\leqq\frac{\varepsilon_0 c^3}{d^2} \quad (z\neq 0)$ *where* $\varepsilon_0$ *is a sufficiently small positive constant,*

( vi ) $abc-cg'(y)-adf(z)\geqq\delta_0>0$ *for all* $y, z$,

( vii ) $g'(y)-g_1(y)\leqq\delta<\frac{2d\delta_0}{ac^2} \qquad (y\in R^1)$,

(viii) $f_1(z)-f(z)\leqq\frac{c\delta}{ad} \qquad (z\in R^1)$,

( ix ) $|p(t,x,y,z,w)|\leqq p_1(t)+p_2(t)(y^2+z^2+w^2)^{\rho/2}+\Delta_1(y^2+z^2+w^2)^{1/2}$ *where* $\rho, \Delta_1$ *are constants such that* $0\leqq\rho<1$, $\Delta_1\geqq 0$ *and the functions* $p_1\geqq 0$, $p_2\geqq 0$,

( x ) $\max_{t\geqq 0} p_i(t)<\infty$ *and* $\int_0^\infty p_i(t)dt<\infty \qquad (i=1,2)$.

*If* $\Delta_1$ *is sufficiently small, then every solution* $x=x(t)$ *of* (1.1) *satisfies*

(2.1) $$x\to 0, \dot{x}\to 0, \ddot{x}\to 0, \dddot{x}\to 0 \qquad \text{as } t\to\infty.$$

**Remark 1.** For the special cases (1) $f(z)=a, p(t,x,y,z,w)\equiv 0$ and (2) $\phi(y,z)=a_2 z$, our result is the same as Theorem 1 in [5] and Theorem 1 in [1] respectively with the weaker condition that

$$xh(x)>0, \quad \int_0^x h(\xi)d\xi\to 0 \quad \text{as } |x|\to\infty \quad \text{than } \frac{h(x)}{x}\geqq a_4>0.$$

For the case $p(t)\not\equiv 0$, the boundedness of the solution of (1.2) is obtained in [5] under the condition that

$$\int_0^\infty |p(t)|dt<\infty.$$

However, the above condition, in fact more weaker conditions (ix) and (x) in our Theorem, guarantees (2.1).

**Remark 2.** The real number $\varepsilon_0$ in the condition (v) is a positive constant defined as

$$\varepsilon_0<\min\left\{\frac{2d\delta_0-\delta ac^2}{2abc(ad+c)}, \frac{d\delta_0}{2abc(ad+c)}, \frac{1}{a}, \frac{d}{c}\right\}.$$

**3. Proof of Theorem.** Consider the function $V(x,y,z,w)$ defined by

$$2V(x, y, z, w)=2\beta\int_0^x h(\xi)d\xi+2\int_0^y g(\eta)d\eta+2\alpha\int_0^z \phi(y, \zeta)d\zeta$$
$$+2\int_0^z f(\zeta)d\zeta+2\beta y\int_0^z f(\zeta)d\zeta+(\beta b-\alpha d)y^2-\beta z^2+\alpha w^2$$
$$+2h(x)y+2\alpha h(x)z+2\alpha z g(y)+2\beta y w+2zw$$

where $\alpha=\dfrac{1}{a}+\varepsilon$, $\beta=\dfrac{d}{c}+\varepsilon$ and $\varepsilon>0$ is a constant to be determined in the detailed proof.

We can show that there is a positive constant $D$ such that

$$V\geqq D\left\{\int_0^x h(\xi)d\xi+y^2+z^2+w^2\right\}.$$

The remainder of the proof is analogous to the proof of Theorem 1 in [1].

## References

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