68. *A Note on the Asymptotic Behavior of the Solutions of* $\dddot{x}+a(t)f(\ddot{x})\dddot{x}+b(t)\phi(\dot{x},\ddot{x})+c(t)g(\dot{x})+d(t)h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x})$

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1. Introduction. In this note we shall be concerned with fourth order non-autonomous differential equations of the form

(1.1) $\ddddot{x}+a(t)f(\dddot{x})\dddot{x}+b(t)\phi(\dot{x},\ddot{x})+c(t)g(\dot{x})+d(t)h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x})$

where a,b,c,d,f,ϕ,g,h and p are continuous real-valued functions depending only on the arguments displayed, and dots indicate differentiation with respect to t.

Many authors (J. O. C. Ezeilo [3], M. Harrow [6], A. S. C. Sinha and R. G. Hoft [10], M. A. Asmussen [1], B. S. Lalli and W. S. Skrapek [8], T. Hara [4], etc. [9]) have studied the stability of the trivial solution of the fourth order autonomous differential equations of the form

(1.2) $$\ddddot{x}+f(\dddot{x})\dddot{x}+\phi(\dot{x},\ddot{x})+g(\dot{x})+h(x)=0$$

and their perturbed equations of the form

(1.3) $$\ddddot{x}+f(\dddot{x})\dddot{x}+\phi(\dot{x},\ddot{x})+g(\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x},\dddot{x}).$$

We shall investigate sufficient conditions under which all solutions of the non-autonomous differential equation (1.1) tend to zero as $t\to\infty$.

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2. Assumptions and theorem. Let us begin by stating the assumptions on the functions appeared in the equation (1.1).

Assumptions.

(I) $a(t), b(t), c(t)$ and $d(t)$ are C^1-functions in $I=[0,\infty)$.

(II) $f(z)$ is a C^1-function in R^1.

(III) The functions $\phi(y,z)$ and $\dfrac{\partial\phi}{\partial y}(y,z)$ are continuous in R^2.

(IV) $g(y)$ is a C^1-function in R^1.

(V) $h(x)$ is a C^1-function in R^1.

(VI) The function $p(t,x,y,z,w)$ is continuous in $I\times R^4$.

Hereafter the following notations are used:

$$g_1(y)=\frac{g(y)}{y}\quad (y\neq 0),\qquad g_1(0)=g'(0),$$

$$f_1(z)=\frac{1}{z}\int_0^z f(\zeta)d\zeta\quad (z\neq 0),\qquad f_1(0)=f(0).$$

Our result is summarized in the following theorem:

Theorem. *Assume that the assumptions* (I)~(VI) *hold and that there exist positive constants such that*

(i) $A \geqq a(t) \geqq a_0 > 0, B \geqq b(t) \geqq b_0 > 0, C \geqq c(t) \geqq c_0 > 0,$
$D \geqq d(t) \geqq d_0 > 0, \quad (t \in I),$

(ii) $f(z) \geqq f_0 > 0 \quad (z \in R^1),$

(iii) $g_1(y) \geqq g_0 > 0 \quad (y \in R^1), \qquad g(0) = 0,$

(iv) $xh(x) > 0 \quad (x \neq 0), \quad H(x) \equiv \int_0^x h(\xi) d\xi \to \infty \qquad$ *as* $|x| \to \infty,$
$h_0 - \frac{a_0 f_0 \delta_0}{2c_0 g_0} \leqq h'(x) \leqq h_0,$

(v) $\phi_y(y, z) \leqq 0, \quad \phi(y, 0) = 0 \qquad$ *in* $R^2,$

(vi) $0 \leqq \frac{\phi(y,z)}{z} - \phi_0 \leqq \frac{\varepsilon_0 c_0^3 g_0^3}{BD^2 h_0^2} \quad (z \neq 0) \qquad$ *where* ε_0 *is a sufficiently small positive constant,*

(vii) $a_0 b_0 c_0 f_0 \phi_0 g_0 - C^2 g_0 g'(y) - A^2 D f_0 h_0 f(z) \geqq \delta_0 > 0$
for all $(y, z) \in R^2,$

(viii) $g'(y) - g_1(y) \leqq \delta < \frac{2Dh_0\delta_0}{Ca_0 f_0 c_0^2 g_0^2},$

(ix) $f_1(z) - f(z) \leqq \frac{Cc_0 g_0 \delta}{Aa_0 f_0 D h_0},$

(x) $\int_0^\infty \{|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\} dt < \infty, \; d'(t) \to 0 \qquad$ *as* $t \to \infty,$

(xi) $|p(t, x, y, z, w)| \leqq p_1(t) + p_2(t)\{H(x) + y^2 + z^2 + w^2\}^{\rho/2} + \Delta(y^2 + z^2 + w^2)^{1/2}$ *where* ρ, Δ *are constants such that* $0 \leqq \rho \leqq 1$, $\Delta \geqq 0$ *and* $p_1(t), p_2(t)$ *are non-negative continuous functions,*

(xii) $\sup_{t \geqq 0} p_i(t) < \infty, \quad \int_0^\infty p_i(t) dt < \infty \qquad (i = 1, 2).$

If Δ *is sufficiently small, then every solution* $x(t)$ *of* (1.1) *is uniform-bounded and satisfies*

(2.1) $\qquad x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0, \quad \dddot{x}(t) \to 0 \qquad$ *as* $t \to \infty.$

Remark. It should be pointed out that in the special case $a(t) = b(t) = c(t) = d(t) = 1$ above Theorem reduces to the author's earlier result [4]. Also our result contains the results in [1] and [6].

The real number ε_0 in the condition (vi) is a positive constant such that

$$\varepsilon_0 < \min \left\{ \frac{AC(2Dh_0\delta_0 - Ca_0 f_0 c_0^2 g_0^2 \delta)}{2a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 D h_0 + Cg_0)}, \frac{ACDh_0\delta_0}{2a_0^2 b_0 c_0^2 f_0 g_0 \phi_0 (Af_0 D h_0 + Cg_0)}, \frac{1}{a_0 f_0}, \frac{Dh_0}{c_0 g_0} \right\}.$$

3. Proof of Theorem. We show the outline of the proof. Consider the function $V(t, x, y, z, w)$ defined by

$$2V(t, x, y, z, w) = 2\beta d(t) \int_0^x h(\xi) d\xi + 2c(t) \int_0^y g(\eta) d\eta$$

$$(3.1)\qquad +2\alpha b(t)\int_0^z \phi(y,\zeta)d\zeta+2a(t)\int_0^z f(\zeta)\zeta d\zeta+2\beta a(t)y\int_0^z f(\zeta)d\zeta$$
$$+\{\beta\phi_0 b(t)-\alpha h_0 d(t)\}y^2-\beta z^2+\alpha w^2+2d(t)h(x)y$$
$$+2\alpha d(t)h(x)z+2\alpha c(t)zg(y)+2\beta yw+2zw$$

where $\alpha=\dfrac{1}{a_0 f_0}+\varepsilon$, $\beta=\dfrac{h_0 D}{c_0 g_0}+\varepsilon$ and ε is a constant to be determined in the detailed proof.

Taking ε to be sufficiently small, we can find positive numbers D_1 and D_2 such that

$$(3.2)\qquad D_1\{H(x)+y^2+z^2+w^2\}\leqq V(t,x,y,z,w)\leqq D_2\{H(x)+y^2+z^2+w^2\}.$$

The remainder of the proof is similar to the latter half of the proof of Theorem 1 in [1]. But details are somewhat complicated, for the equation (1.1) is non-autonomous. The detailed proof will be published later in some journal.

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