113. A Note on Cauchy Problems of Semi-linear Equations and Semi-groups in Banach Spaces

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§1. Introduction. Let X be a real Banach space with the norm $\| \ \|$. An operator B in X is said to be *accretive* if

(1.1) $\|(I+\lambda B)x-(I+\lambda B)y\| \geq \|x-y\|$ for $x, y \in D(B)$ and $\lambda > 0$.

It is known that B is accretive if and only if for any $x, y \in D(B)$ there exists $f \in F(x-y)$ such that $(Bx-By, f) \geq 0$, where F is the duality map of X, i.e., $F(x)=\{x^* \in X^*; (x, x^*)=\|x\|^2=\|x^*\|^2\}$ for $x \in X$. If B is accretive and $R(I+\lambda B)=X$ for all $\lambda>0$, we say that B is *m-accretive*.

Let A be a linear m-accretive operator in X with dense domain and let B be a nonlinear accretive operator in X. Recently G. Webb [4] proved that, under some additional assumptions on A and B, for all $x \in X$ and $t \geq 0$

(1.2) $$U(t)x=\lim_{n\to\infty}((I+(t/n)B)^{-1}(I+(t/n)A)^{-1})^n x$$

exists and $\{U(t); t\geq 0\}$ is a contraction semi-group on X. By a *contraction semi-group* on C, where C is a subset of X, we mean a family $\{U(t); t\geq 0\}$ of operators $U(t): C\to C$ satisfying the following conditions: (1) $U(t)U(s)=U(t+s)$ for $t, s\geq 0$; (2) $\lim_{t\to 0+}U(t)x=U(0)x=x$ for $x\in C$; (3) $U(t)$, $t\geq 0$, are contractions on C, i.e., $\|U(t)x-U(t)y\|\leq\|x-y\|$ for $x, y\in C$, $t\geq 0$.

In this paper, we shall study how the semi-group $\{U(t); t\geq 0\}$ given by (1.2) is related to the strong solution of the following Cauchy problem

(1.3) $$du/dt+(A+B)u=0, \qquad u(0)=x\ (\in X).$$

Now we give the precise definition of strong solution of the Cauchy problem (1.3).

Definition 1.1. A function $u: [0, \infty)\to X$ is a *strong solution* of (1.3) if u is Lipschitz continuous on $[0, \infty)$, $u(0)=x$, u is strongly differentiable almost everywhere and

(1.4) $$du(t)/dt+(A+B)u(t)=0 \qquad \text{for a.a. } t\in[0, \infty).$$

It follows easily from the accretiveness of $A+B$ that the Cauchy problem has at most one strong solution.

Our results are stated as follows; and the proofs are given in §2.

Theorem 1.1. *Suppose that A is a linear m-accretive operator in X with dense domain, B is a nonlinear accretive operator in X and D is a subset of $D(A)\cap D(B)$ satisfying $(I+\lambda B)^{-1}(I+\lambda A)^{-1}(D)\subset D$ for $\lambda>0$. Let $u: [0, \infty)\to X$ be a strong solution of the Cauchy problem* (1.3) *with* *the initial value* $u(0)=x\in D$, *and assume that for any* $T>0$ *there exists a constant* $M_T>0$ *such that*

(1.5) $$\|ABu(t)\|\le M_T \qquad \textit{for a.a. } t\in[0,T].$$

Then we have

(1.6) $$u(t)=\lim_{n\to\infty}((I+(t/n)B)^{-1}(I+(t/n)A)^{-1})^n x \qquad \textit{for } t\ge 0.$$

Remark. By applying this theorem with $Ax=0$ for $x\in X$ and $D=D(B)$, we can obtain the following result due to Brezis and Pazy [1]: *Let* B *be a nonlinear accretive operator in* X *such that* $R(I+\lambda B)\supset D(B)$ *for* $\lambda>0$. *If* $u:[0,\infty)\to X$ *is a strong solution of* $du/dt+Bu=0$, $u(0)=x\in D(B)$, *then*

$$u(t)=\lim_{n\to\infty}(I+(t/n)B)^{-n}x \qquad \textit{for } t\ge 0.$$

Theorem 1.2. *Suppose that* A *is a linear m-accretive operator in* X *with dense domain,* B *is a nonlinear closed accretive operator in* X, $B0=0$, *and* D *is a subset of* $D(A)\cap D(B)$, $D\ni 0$, *such that*

(1.7) $$(I+\lambda B)^{-1}(I+\lambda A)^{-1}(\bar{D})\subset D \qquad \textit{for } \lambda>0;$$

(1.8) *there is a normed space* $Y\supset D$ *with the norm* $\|\ \|_0$ *such that* $(I+\lambda A)^{-1}$ *is* $\|\ \|_0$*-constraction on* D *and* $(I+\lambda B)^{-1}$ *is* $\|\ \|_0$*-constraction on* $\bar{D}\cap Y$ *for* $\lambda>0$;

(1.9) *there is an increasing function* $L:[0,\infty)\to(0,\infty)$ *such that for all* $x\in D\cap D(AB)$, $\|ABx\|\le L(\|x\|_0)\cdot\|Ax\|$.

Then, for each $x\in\bar{D}$ *and* $t\ge 0$

(1.10) $$U(t)x=\lim_{n\to\infty}((I+(t/n)B)^{-1}(I+(t/n)A)^{-1})^n x$$

exists and $\{U(t);t\ge 0\}$ *is a contraction semi-group on* $\bar{D}$.

In addition to the assumptions above, suppose that the Banach space $D(A)$ *with the graph norm* $\|\ \|_A$ *is continuously embedded into* Y.

If $x\in\bar{D}$ *and there exists the strong derivative* $(d/dt)U(t)x$ *at some point* $t_0>0$ *such that* $U(t_0)x\in D(A)$, *then* $U(t_0)x\in D(B)$ *and* $(A+B)U(t_0)x=-(d/dt)U(t)x|_{t=t_0}$.

Finally we have the following existence theorem for the solution of the Cauchy problem (1.3) in reflexive Banach space.

Theorem 1.3. *Let* X *be a reflexive Banach space and suppose the assumptions in Theorem* 1.2 *are satisfied. Then for each* $x\in D$, $u(t)=U(t)x$ *given by* (1.10) *is the unique strong solution of the Cauchy problem* (1.3).

§2. Proofs of Theorems. We start from the

Proof of Theorem 1.1. Let $J_\lambda=(I+\lambda B)^{-1}(I+\lambda A)^{-1}$ for $\lambda>0$, then J_λ is single-valued contraction and $J_\lambda(D)\subset D$. Let $\{\varepsilon_n\}$ be a positive sequence such that $\varepsilon_n\to 0$ as $n\to\infty$ and define step functions $u_n(t)$ on $[0,\infty)$ by

(2.1) $$u_n(t)=J_{\varepsilon_n}^{[t/\varepsilon_n]}x \qquad \text{for } t\ge 0.$$

If we set

(2.2) $$v_n(t)=u_n(j\varepsilon_n)+\varepsilon_n^{-1}(t-j\varepsilon_n)(J_{\varepsilon_n}^{j+1}x-J_{\varepsilon_n}^{j}x)$$

for $j\varepsilon_n\le t\le(j+1)\varepsilon_n$, $j=0,1,2,\cdots$, then $v_n(t)$ is differentiable on

$(j\varepsilon_n, (j+1)\varepsilon_n)$, $j=0,1,2,\cdots$, and we have

$$(d/dt)v_n(t)=\varepsilon_n^{-1}(J_{\varepsilon_n}^{j+1}x-J_{\varepsilon_n}^{j}x) \qquad \text{for } j\varepsilon_n<t<(j+1)\varepsilon_n.$$

Therefore

$$\begin{aligned}\|(d/dt)v_n(t)\| &= \varepsilon_n^{-1}\|J_{\varepsilon_n}^{j+1}x-J_{\varepsilon_n}^{j}x\| \le \varepsilon_n^{-1}\|J_{\varepsilon_n}x-x\| \\ &\le \|Ax\|+\|Bx\| \qquad \text{for a.a. } t\ge 0,\end{aligned} \tag{2.3}$$

where we used the following inequality: for $\lambda>0$, $x\in D$

$$\|J_\lambda x-x\|\le\|(I+\lambda A)^{-1}x-(I+\lambda B)x\|\le\lambda(\|Ax\|+\|Bx\|).$$

Using this inequality we also have

$$\|v_n(t)-u_n(t)\|\le\varepsilon_n(\|Ax\|+\|Bx\|) \qquad \text{for } t\ge 0. \tag{2.4}$$

By the definition, we have

$$u_n(t)=\begin{cases}J_{\varepsilon_n}u_n(t-\varepsilon_n) & \text{for } t\ge\varepsilon_n,\\ x & \text{for } 0\le t\le\varepsilon_n.\end{cases} \tag{2.5}$$

We extend (the strong solution) $u(t)$ as x for $t<0$ and put

$$g_n(t)=\varepsilon_n^{-1}(u(t)-u(t-\varepsilon_n))-(d/dt)u(t) \qquad \text{for a.a. } t\ge 0. \tag{2.6}$$

Then we have

$$\varepsilon_n^{-1}(u(t)-u(t-\varepsilon_n))+(A+B)u(t)=g_n(t)$$

or

$$u(t)=J_{\varepsilon_n}(u(t-\varepsilon_n)+\varepsilon_n g_n(t)+\varepsilon_n^2 ABu(t)) \qquad \text{for a.a. } t\ge 0. \tag{2.7}$$

Let $T>0$ be arbitrarily fixed. Then it follows from (2.5) and (2.6) that for a.a. $t\in[\varepsilon_n, T]$,

$$\begin{aligned}\|u_n(t)-u(t)\| &\le \|u(t-\varepsilon_n)-u_n(t-\varepsilon_n)+\varepsilon_n g_n(t)+\varepsilon_n^2 ABu(t)\| \\ &\le \|u(t-\varepsilon_n)-u_n(t-\varepsilon_n)\|+\varepsilon_n\|g_n(t)\|+\varepsilon_n^2 M_T.\end{aligned}$$

Integrating this inequality over $[\varepsilon_n, \theta]$ with $\varepsilon_n\le\theta<T$, we have

$$\begin{aligned}\int_{\varepsilon_n}^{\theta}\|u_n(s)-u(s)\|ds &\le \int_{\varepsilon_n}^{\theta}\|u_n(s-\varepsilon_n)-u(s-\varepsilon_n)\|ds \\ &\quad +\varepsilon_n\int_{\varepsilon_n}^{\theta}\|g_n(s)\|ds+\varepsilon_n^2 TM_T\end{aligned}$$

and hence

$$\int_{\theta-\varepsilon_n}^{\theta}\|u_n(s)-u(s)\|ds\le\int_0^{\varepsilon_n}\|x-u(s)\|ds+\varepsilon_n\int_0^T\|g_n(s)\|ds+\varepsilon_n^2 TM_T.$$

Adding these inequalities for $\theta=\varepsilon_n, 2\varepsilon_n, \cdots, [T/\varepsilon_n]\varepsilon_n$, we obtain

$$\begin{aligned}&\int_0^{[T/\varepsilon_n]\varepsilon_n}\|u_n(s)-u(s)\|ds \\ &\le T\left(\varepsilon_n^{-1}\int_0^{\varepsilon_n}\|x-u(s)\|ds+\int_0^T\|g_n(s)\|ds+\varepsilon_n TM_T\right).\end{aligned} \tag{2.8}$$

Since $g_n(t)\to 0$ at a.a. $t\ge 0$ as $n\to\infty$ and $\|g_n(t)\|\le 2M$ for a.a. $t\ge 0$, where M is a Lipschitz constant for $u(t)$, we have $\int_0^T\|g_n(t)\|dt\to 0$ as $n\to\infty$. Therefore (2.8) implies $\lim_{n\to\infty}\int_0^T\|u_n(t)-u(t)\|dt=0$. Combining this with the inequality (2.4), we have $\lim_{n\to\infty}\int_0^T\|v_n(t)-u(t)\|dt=0$. Since

$$\begin{aligned}(d/dt)\|v_n(t)-u(t)\| &\le \|(d/dt)(v_n(t)-u(t))\| \\ &\le \|(d/dt)v_n(t)\|+\|(d/dt)u(t)\|\le\|Ax\|+\|Bx\|+M,\end{aligned}$$

we have

$$\frac{1}{2}(d/dt)\|v_n(t)-u(t)\|^2=\|v_n(t)-u(t)\|\cdot(d/dt)\|v_n(t)-u(t)\|$$
$$\leq(\|Ax\|+\|Bx\|+M)\cdot\|v_n(t)-u(t)\| \qquad \text{for a.a. } t\geq 0$$

and hence

$$\|v_n(t)-u(t)\|^2\leq 2(\|Ax\|+\|Bx\|+M)\int_0^T\|v_n(s)-u(s)\|\,ds$$

for $t\in[0,T]$. Consequently $u(t)=\lim_{n\to\infty}v_n(t)=\lim_{n\to\infty}u_n(t)$, uniformly on $[0,T]$. If we put $\varepsilon_n=t/n$ in particular, we obtain (1.6). Q.E.D.

For the proof of Theorem 1.2, we prepare some lemmas. First we state the following lemma without proof. (See Webb [4] for the proof.)

Lemma 2.1. *Suppose that A is a linear m-accretive operator in X with dense domain, B is a nonlinear operator in X, $B0=0$, and D is a subset of $D(A)\cap D(B)$, $D\ni 0$, such that* (1.7), (1.8) *and* (1.9) *in Theorem* 1.2 *are satisfied.*

If we put $J_\lambda=(I+\lambda B)^{-1}(I+\lambda A)^{-1}$ for $\lambda>0$, then for each $x\in D$, $0<\lambda<L(\|x\|_0)^{-1}$ and $n=1,2,\cdots$, we have that $J_\lambda^n x\in D\cap D(AB)$,

$$\|AJ_\lambda^n x\|\leq(1-\lambda L(\|x\|_0))^{-n}\|Ax\| \tag{2.9}$$

and

$$\|ABJ_\lambda^n x\|\leq L(\|x\|_0)\cdot(1-\lambda L(\|x\|_0))^{-n}\|Ax\|. \tag{2.10}$$

Moreover, for $x\in D$, the integer $n\geq m\geq 1$ and $\lambda\geq\mu>0$,

$$\begin{aligned}\|J_\lambda^n x-J_\mu^m x\|\leq{}&[((n\mu-m\lambda)^2+n\mu(\lambda-\mu))^{1/2}+(m\lambda(\lambda-\mu)\\&+(m\lambda-n\mu)^2)^{1/2}](\|Ax\|+\|Bx\|)+n\mu(\lambda-\mu)\cdot\max_{1\leq k\leq n}\|ABJ_\lambda^k x\|.\end{aligned} \tag{2.11}$$

And then for each $x\in\bar{D}$ and $t\geq 0$

$$U(t)x=\lim_{n\to\infty}((I+(t/n)B)^{-1}(I+(t/n)A)^{-1})^n x \tag{2.12}$$

exists and $\{U(t);t\geq 0\}$ is a contraction semi-group on $\bar{D}$.

In the following let $U(t)$ be given by (2.12). Next we present the following useful lemma.

Lemma 2.2. *Let the hypothesis of Lemma* 2.1 *be satisfied. Then we have for $x\in\bar{D}$ and $x_0\in D$*

$$\sup_{\zeta^*\in F(x-x_0)}\limsup_{t\to 0+}\left(\frac{U(t)x-x}{t},\zeta^*\right)\leq\langle(A+B)x_0,x_0-x\rangle_s, \tag{2.13}$$

where $\langle x,y\rangle_s=\sup\{(x,\eta^*);\eta^*\in F(y)\}$ for $x,y\in X$.

Proof. It follows from Lemma 2.1 that

$$U(t)x=\lim_{\varepsilon\to 0+}((I+\varepsilon B)^{-1}(I+\varepsilon A)^{-1})^{[t/\varepsilon]}x$$

for $t\geq 0$ and $x\in\bar{D}$. At first, let $x\in D$ and put

$$y_{\lambda,k}=\lambda^{-1}(J_\lambda^{k-1}x-J_\lambda^k x) \quad \text{for } \lambda>0 \quad\text{and}\quad k=1,2,\cdots.$$

Then $J_\lambda^k x\in D\cap D(AB)$ and

$$y_{\lambda,k}=(A+B)J_\lambda^k x+\lambda ABJ_\lambda^k x. \tag{2.14}$$

Since $A+B$ is accretive, there is a $\eta^*\in F(x_0-J_\lambda^k x)$ such that

$$(y_{\lambda,k}-(A+B)x_0-\lambda ABJ_\lambda^k x,\eta^*)\leq 0.$$

Hence

$$\|x_0-J_\lambda^k x\|^2-\|x_0-J_\lambda^{k-1}x\|^2$$

$$\leq 2(\|x_0-J_\lambda^k x\|^2-\|x_0-J_\lambda^{k-1}x\|\cdot\|x_0-J_\lambda^k x\|)$$
$$\leq 2\lambda(y_{\lambda,k},\eta^*)\leq 2\lambda((A+B)x_0,\eta^*)+2\lambda^2(ABJ_\lambda^k x,\eta^*)$$
$$\leq 2\lambda\langle (A+B)x_0, x_0-J_\lambda^k x\rangle_s+2\lambda^2\|ABJ_\lambda^k x\|\cdot\|x_0-J_\lambda^k x\|$$
$$\leq 2\int_{k\lambda}^{(k+1)\lambda}\langle (A+B)x_0, x_0-J_\lambda^{[s/\lambda]}x\rangle_s ds+2\lambda^2\|ABJ_\lambda^k x\|\cdot(\|x_0\|+\|x\|).$$

Let $t\geq\lambda$ and add the above inequalities for $k=1,2,\cdots,[t/\lambda]$. Then we have

$$\|x_0-J_\lambda^{[t/\lambda]}x\|^2-\|x_0-x\|^2\leq 2\int_{\lambda}^{([t/\lambda]+1)\lambda}\langle (A+B)x_0, x_0-J_\lambda^{[s/\lambda]}x\rangle_s ds$$
$$+2\lambda^2\left(\textstyle\sum_{k=1}^{[t/\lambda]}\|ABJ_\lambda^k x\|\right)\cdot(\|x_0\|+\|x\|).$$

Since $\langle\ ,\ \rangle_s : X\times X\to(-\infty,\infty)$ is upper semi-continuous (see [2]) and $\lambda^2\sum_{k=1}^{[t/\lambda]}\|ABJ_\lambda^k x\|=o(\lambda)$ as $\lambda\to 0+$ by (2.10), by taking the limit superior as $\lambda\to 0+$ in this inequality, we obtain

$$\text{(2.15)}\quad \|U(t)x-x_0\|^2-\|x_0-x\|^2\leq 2\int_0^t\langle (A+B)x_0, x_0-U(s)x\rangle_s ds \quad \text{for } t\geq 0.$$

Noting $\|U(t)x-x_0\|^2-\|x-x_0\|^2\geq 2(U(t)x-x,\zeta^*)$ for any $\zeta^*\in F(x-x_0)$, (2.15) yields

$$\text{(2.16)}\quad (U(t)x-x,\zeta^*)\leq\int_0^t\langle (A+B)x_0, x_0-U(s)\rangle_s ds$$

for $t\geq 0$. It is easy to see that (2.15) and (2.16) remain true for $x\in\bar{D}$. Dividing (2.16) by $t>0$ and taking the limit superior as $t\to 0+$, we have the desired inequality (2.13). Q.E.D.

Proof of Theorem 1.2. The first part of theorem has been already shown. We shall prove the second part. If we set $y=(d/dt)U(t)x|_{t=t_0}$, we can write

$$U(t_0-\lambda)x=U(t_0)x-\lambda y+o(\lambda) \qquad \text{as } \lambda\to 0+.$$

Since $\lambda BJ_\lambda U(t_0-\lambda)x=(I+\lambda A)^{-1}U(t_0-\lambda)x-J_\lambda U(t_0-\lambda)x$ and $D(A)$ is a linear space, $x_\lambda\equiv J_\lambda U(t_0-\lambda)x\in D(AB)\cap D(A)$ for $\lambda>0$. Therefore we have

$$\text{(2.17)}\quad x_\lambda+\lambda(A+B)x_\lambda+\lambda^2 ABx_\lambda=U(t_0)x-\lambda y+o(\lambda).$$

We want to show that ABx_λ is bounded as $\lambda\to 0+$. We first note that

$$\|A_\lambda U(t_0-\lambda)x\|\leq\|A_\lambda U(t_0)x\|+\|A_\lambda U(t_0)x-A_\lambda U(t_0-\lambda)x\|$$
$$\leq\|AU(t_0)x\|+2\lambda^{-1}\|U(t_0)x-U(t_0-\lambda)x\|=O(1)$$

as $\lambda\to 0+$, where $A_\lambda=A(I+\lambda A)^{-1}$ for $\lambda>0$. Since the Banach space $D(A)$ with the graph norm is contineously embedded into Y, furthermore we have

$$\|x_\lambda\|_0\leq\|(I+\lambda A)^{-1}U(t_0-\lambda)x\|_0\leq C\,\|(I+\lambda A)^{-1}U(t_0-\lambda)x\|_A$$
$$\leq C(\|U(t_0-\lambda)x\|+\|A_\lambda U(t_0-\lambda)x\|)=O(1)$$

as $\lambda\to 0+$, where C is a positive constant. By (1.9), we have

$$\|Ax_\lambda\|\leq\|A(I+\lambda A)^{-1}U(t_0-\lambda)x\|+\|ABJ_\lambda U(t_0-\lambda)x\|$$
$$\leq\|A_\lambda U(t_0-\lambda)x\|+L(\|x_\lambda\|_0)\cdot\|Ax_\lambda\|$$

and then we have for sufficiently small $\lambda>0$

$$\|Ax_\lambda\|\leq(1-\lambda L(\|x_\lambda\|_0))^{-1}\|A_\lambda U(t_0-\lambda)x\|.$$

Therefore we obtain

$$\|ABx_\lambda\|\leq L(\|x_\lambda\|_0)\,\|Ax_\lambda\|\leq L(\|x_\lambda\|_0)\cdot(1-\lambda L(\|x_\lambda\|_0))^{-1}\|A_\lambda U(t_0-\lambda)x\|=O(1)$$

as $\lambda\to 0+$. Accordingly we have by (2.17)

(2.18) $\qquad x_\lambda+\lambda(A+B)x_\lambda=U(t_0)x-\lambda y+o(\lambda) \qquad$ as $\lambda\to 0+$.

Hence the standard argument implies that $\lambda^{-1}\|x_\lambda-U(t_0)x\|\to 0$ and $\|(A+B)x_\lambda+y\|\to 0$ as $\lambda\to 0+$. (See the proof of Theorem II of [2].) We can rewrite (2.17) in the fashion:

$$\begin{aligned} x_\lambda+\lambda Bx_\lambda &= (I+\lambda A)^{-1}(U(t_0)x-\lambda y+o(\lambda)) \\ &= U(t_0)x-\lambda A(I+\lambda A)^{-1}U(t_0)x-(I+\lambda A)^{-1}y+o(\lambda) \end{aligned}$$

or

$$(I+\lambda A)^{-1}AU(t_0)x+Bx_\lambda+(I+\lambda A)^{-1}y=\lambda^{-1}(U(t_0)x-x_\lambda)+o(1)$$

as $\lambda\to 0+$. Therefore the closedness of B implies that $U(t_0)x\in D(B)$ and $(A+B)U(t_0)x=-y$. Q.E.D.

Proof of Theorem 1.3. It is easily shown that $u(t)=U(t)x$ is Lipschitz continuous on $[0,\infty)$ by Lemma 2.1. Therefore it follows from the reflexivity of X that $u(t)$ is strongly differentiable almost everywhere on $(0,\infty)$.

We now show that $U(t)x\in D(A)$ for all $t>0$. In fact, for each $t>0$, $\lim_{n\to\infty} J^n_{t/n}x=U(t)x$ and $AJ^n_{t/n}x$ is bounded as $n\to\infty$ by (2.9). So the weak closedness of A implies that $U(t)x\in D(A)$. (The weak closedness of A means that $x_n\in D(A)$, $x_n\rightharpoonup x$ (weak convergence) and $Ax_n\rightharpoonup y$ imply $x\in D(A)$ and $y=Ax$.) Consequently, by Theorem 1.2, $u(t)$ is a strong solution of the Cauchy problem (1.3). Q.E.D.

Remark. If we suppose in addition in Theorem 1.3 that X has a uniformly convex dual and B is m-accretive, then the solution $u(t)$ of Theorem 1.3 satisfies the condition (1.5) of Theorem 1.1. In fact, let $T>0$ be arbitrarily fixed. Then $\|AJ^n_{t/n}x\|=O(1)$ as $n\to\infty$ uniformly on $[0,T]$, and so is $\|ABJ^n_{t/n}\|$ by (2.10). Therefore by (2.14)

$$\begin{aligned} \|BJ^n_{t/n}x\| &\le \|AJ^n_{t/n}x\|+(t/n)\|ABJ^n_{t/n}x\|+\|y_{t/n,n}\| \\ &\le \|AJ^n_{t/n}x\|+(t/n)\|ABJ^n_{t/n}x\|+\|Ax\|+\|Bx\|=O(1) \end{aligned}$$

as $n\to\infty$, uniformly on $[0,T]$. Consequently, the weak closedness of A and the demi-closedness of B imply that $U(t)x\in D(AB)$ and $\|ABU(t)x\|$ is bounded on $[0,T]$. Q.E.D.

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