44. On a Parametrix in Some Weak Sense of a First Order Linear Partial Differential Operator with Two Independent Variables

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Introduction. Let $L=\partial/\partial t+i\phi(x)\sigma(t)\partial/\partial x$ be a first order linear partial differential operator with two independent variables in an open rectangle $\Omega=(a,b)\times(\alpha,\beta)\subset R_x^1\times R_t^1$, $-\infty\leqq a<b\leqq+\infty$, $-\infty\leqq\alpha<0<\beta\leqq+\infty$. In this paper we construct a parametrix of L in some weak sense and consider the regularity of the solution of the equation,

(0.1) $$Lu=f \quad \text{in } \Omega,$$

under the assumptions that

(0.2) $\phi\in C^\infty((a,b))$, and all derivatives of ϕ are bounded,

(0.3) $\sigma\in C^\infty((\alpha,\beta))$, $\sigma(t)\geqq 0$ in (α,β), and zeros of σ are all of finite order.

Equation (0.1) is locally solvable in Ω under these assumptions (cf. [1], [4]), but is not hypoelliptic in general (cf. [6]). In § 4 it will be seen how the regularity, with respect to t, of the solution u of (0.1) increases.

§ 1. Outline of the construction of a parametrix. We consider the solution of the form

(1.1) $$u(x,t)=\frac{1}{2\pi i}\int\exp\left(i\xi\int_0^t\sigma(s)ds\right)v(x,\xi)d\xi.$$

Calculating formally, we have

(1.2) $$Lu=\frac{\sigma(t)}{2\pi}\int\exp\left(i\xi\int_0^t\sigma(s)ds\right)(\xi v(x,\xi)+\phi(x)\partial/\partial x v(x,\xi))d\xi.$$

Remark that if $\sigma(t)>0$ in (α,β)

(1.3) $$g(t)=\frac{\sigma(t)}{2\pi}\int\exp\left(i\xi\int_0^t\sigma(s)ds\right)\left(\int\exp\left(-i\xi\int_0^{t'}\sigma(s)ds\right)g(t')dt'\right)d\xi$$

for every $g\in C_0^\infty((\alpha,\beta))$. Then, we can expect that when the solution v of the equation

(1.4) $$\xi v(x,\xi)+\phi(x)\partial/\partial x v(x,\xi)=\int\exp\left(-i\xi\int_0^{t'}\sigma(s)ds\right)f(x,t')dt'$$

is substituted in the right-hand side of (1.1) $u(x,t)$ will give a solution of (0.1).

§ 2. Preliminary lemmas. We state two lemmas for the construction of a parametrix of L without proof.

Lemma 2.1. *Let ϕ satisfy* (0.2). *We consider the equation*

(2.1) $\xi v(x)+\phi(x)d/dxv(x)=f(x) \quad in\ (a, b)$

with ξ *a real parameter. Then, for every positive integer* j*, there exists a constant* $C_j>0$*, such that for* $|\xi|>C_j$ *we can find a linear mapping* $S_\xi: C_0^{j+1}((a, b))\to C^j((a, b))$ *having the following properties:*

(2.2) $\xi S_\xi f+\phi d/dxS_\xi f=f \quad in\ (a, b),$

(2.3) $\phi d/dxS_\xi f=S_\xi(\phi d/dxf),$

(2.4) *When* $S_\xi f$ *is considered as a function of* (x, ξ), $\partial^p/\partial x^p S_\xi f$ *is infinitely differentiable with respect to* ξ *in* $|\xi|>C_j$ *for* $0\leqq p\leqq j$*, and continuous in* $(a, b)\times\{\xi\,|\,|\xi|>C_j\}$.

Furthermore, the following two inequalities hold with a constant C *independent of* f*, for every non negative integer* N:

(2.4.1) $|\partial^N/\partial\xi^N\partial^p/\partial x^p S_\xi f(x)|\leqq C(1+|\xi|)^{-N-1}\sup_{a<x<b}\sum_{0\leqq l\leqq p}|\partial^l/\partial x^l f(x)|$

(2.4.2) $\int_a^b|\partial^N/\partial\xi^N\partial^p/\partial x^p S_\xi f(x)|^2dx\leqq C(1+|\xi|)^{-2N-2}\int_a^b\sum_{0\leqq l\leqq p}|d^l/dx^l f(x)|^2dx$

for $f\in C_0^{j+1}((a, b))$, $|\xi|>C_j$*, and* $0\leqq p\leqq j$.

Proof is omitted, but we give the explicit expression of $S_\xi f$.

Set $M=\{x\in(a, b)\,|\,\phi(x)=0\}$ and decompose $(a, b)\backslash M$ into a disjoint union of open intervals $(a_\mu, b_\mu)_{\mu\in\Lambda}$. We define $S_\xi f$ in the form,

(2.5) $$S_\xi f(x)=\begin{cases}\dfrac{1}{\xi}f(x) & (x\in M)\\ \displaystyle\int_{a_\mu}^x k(x, y, \xi)\frac{1}{\phi(y)}f(y)dy & (\phi, \xi \text{ have the same sign, } x\in I_\mu)\\ \displaystyle-\int_x^{b_\mu} k(x, y, \xi)\frac{1}{\phi(y)}f(y)dy & (\text{otherwise})\end{cases}$$

where $k(x, y, \xi)=\exp\left(\xi\int_x^y\frac{1}{\phi(s)}ds\right)$, and $I_\mu=(a_\mu, b_\mu)$.

Now we introduce some notations. For every $f\in L^1((\alpha, \beta))$ we define $Tf(\xi)$ as follows.

(2.6) $$Tf(\xi)=\int_\alpha^\beta\exp\left(-i\xi\int_0^t\sigma(s)ds\right)f(t)dt.$$

For $\tilde f\in L^1(R_\xi^1)$ we define

(2.7) $$\tilde T\tilde f(t)=\int\exp\left(i\xi\int_0^t\sigma(s)ds\right)\tilde f(\xi)d\xi \qquad \alpha<t<\beta.$$

Lemma 2.2. i) *Let* K *be any compact subset of* (α, β). *For* $\delta>0$*, we have with a constant* C *depending only on* K *and* δ

(2.8) $$|Tf(\xi)|^2\leqq C(1+|\xi|)^{2\delta}\int|\Lambda^{-\delta}f(t)|^2dt$$

where $f\in C_{0,K}^\infty((\alpha, \beta))=\{g\in C_0^\infty((\alpha, \beta))\,|\,\mathrm{supp}\, g\subset K\}$, $|\xi|>1$*, and* $\Lambda^{-\delta}$ *is the pseudo-differential operator with symbol* $(1+|\xi|^2)^{-\delta/2}$.

ii) *Denoting by* l_K *the maximum of the orders of zeros of* σ *in* K*, we have with a constant* C *depending only on* K

(2.9) $$|Tf(\xi)|\leqq C(1+|\xi|)^{-1/(l_K+1)}\sup(|f(t)|+|f'(t)|)$$
for $f\in C_{0,K}^\infty((\alpha, \beta))$*, and* $|\xi|>1$.

iii) *With the same* K, l_K, *and* C *as in* ii) *we have*

(2.10) $$\int_K |\tilde{T}\tilde{f}(t)|^2 dt \leqq C \int |\tilde{f}(\xi)|^2 (1+|\xi|)^{l_K/(l_K+1)} d\xi \qquad \tilde{f} \in L^1(R^1_\xi), |\xi|>1.$$

§ 3. Construction of a parametrix. We introduce some notations

$$H_{r,s} = \left\{ f \in \mathcal{S}'(R^1_x \times R^1_t) \,\middle|\, \|f\|^2_{r,s} = \iint (1+|\xi|^2)^r (1+|\tau|^2)^s |f(\xi,\tau)|^2 d\xi d\tau < +\infty \right\},$$

$H^{\mathrm{loc}}_{r,s}(\Omega) = \{f \in \mathcal{D}'(\Omega) \mid \omega f \in H_{r,s}$ for every $\omega \in C^\infty_0(\Omega)\}$,

$H^0_{r,s}(\Omega) = \mathcal{E}'(\Omega) \cap H_{r,s}$,

$H^0_{r,s,K}(\Omega) = \{f \in H^0_{r,s}(\Omega) \mid t$-projection of supp $f \subset K \Subset (\alpha, \beta)\}$ where r, s are any real numbers.

Theorem 3.1. *Let* L *and* Ω *be as in* § 0, *and assume that* (0.2) *and* (0.3) *hold. Then, for every positive integer* j, *there exist linear mappings* E_j, R_j, *and* R'_j

$$\left.\begin{array}{lll} (3.1) & E_j : H^0_{0,0}(\Omega) \to H^{\mathrm{loc}}_{0,0}(\Omega) \\ (3.2) & R_j : H^0_{r,s}(\Omega) \to H^{\mathrm{loc}}_{r,\tilde{s}}(\Omega) \\ (3.3) & R'_j : H^0_{r,s}(\Omega) \to H^{\mathrm{loc}}_{r,\tilde{s}}(\Omega) \end{array}\right\} \text{for any real numbers } r, s, \tilde{s}$$

having the following properties:

(3.4) $$LE_j f = f + R_j f \quad \text{in } \Omega \quad f \in H^0_{0,0}(\Omega).$$

(3.5) $E_j Lf = f + R'_j f$ *in* Ω *for* $f \in H^0_{0,0}(\Omega)$ *such that* $Lf \in H^0_{0,0}(\Omega)$.

(3.6) *Take any* $\omega \in C^\infty_0(\Omega)$ *and denote by* l_ω *the maximum of the orders of zeros of* σ *in the* t*-projection of* supp ω. *For* $0<\delta<\frac{1}{2}(1+l_\omega)^{-1}$, *and any compact set* K *in* (α, β) *we have, with a constant* C *independent of* f

$$\|\omega\partial^p/\partial x^p E_j f\|_{0,0} \leqq C \|f\|_{p,-\delta} \quad f \in H^0_{0,0,K}(\Omega), \quad 0 \leqq p \leqq j.$$

(3.7) *Let* K *and* ω *be as in* (3.6), *then we have with a constant* C *independent of* f

$$\left.\begin{array}{l} \|\omega R_j f\|_{r,\tilde{s}} \leqq C \|f\|_{r,s} \\ \|\omega R'_j f\|_{r,\tilde{s}} \leqq C \|f\|_{r,s} \end{array}\right\} f \in H^0_{r,s,K}(\Omega).$$

Proof. We define E_j, R_j, and R'_j only for $f \in C^\infty_0(\Omega)$. The extension to the general f can be performed using the approximation by mollifier. Choose a function $\chi_j(\xi) \in C^\infty(R^1_\xi)$ such that $\chi_j(\xi)=0$ $(|\xi| \leqq 2C_j+1)$, and $\chi_j(\xi)=1$ $(|\xi| \geqq 3C_j+1)$, where C_j is the constant appearing in Lemma 2.1. From now on in this proof we drop the subscript j. Now define the operators U and E by the formula

(3.8) $$Uf(x,\xi) = \chi(\xi) S_\xi \left(\int \exp\left(-i\xi \int_0^{t'} \sigma(s)ds\right) f(\cdot, t')dt' \right)(x)$$

(3.9) $$Ef(x,t) = \frac{1}{2\pi i} \int \exp\left(i\xi \int_0^t \sigma(s)ds\right) Uf(x,\xi)d\xi$$

where $f \in C^\infty_0(\Omega)$.

From Lemma 2.1 (2.4.1) and Lemma 2.2 (2.9) we see that (3.9) is well defined, and Ef is continuously differentiable with respect to x up to the order j. Furthermore we can write

$$(3.10) \quad \partial^p/\partial x^p Ef(x,t)=\frac{1}{2\pi i}\int \exp\left(i\xi\int_0^t \sigma(s)ds\right)\partial^p/\partial x^p Uf(x,\xi)d\xi \qquad (0\leqq p\leqq j).$$

Applying Lemma 2.2 (2.10), Lemma 2.1 (2.4.2), and Lemma 2.2 (2.8) successively to (3.10), we obtain (3.6) for $f\in C_0^\infty(\Omega)$. On the other hand, when f vanishes near zeros of σ, Ef is continuously differentiable with respect to t also, and we can write using Lemma 2.1 (2.2) and Fourier inversion formula

$$(3.11) \quad \begin{aligned} LEf(x,t)&=f(x,t)\\ &+\frac{\sigma(t)}{2\pi}\iint \exp\left(i\xi\int_{t'}^t \sigma(s)ds\right)(\chi(\xi)-1)f(x,t')dt'd\xi. \end{aligned}$$

For a general $f\in C_0^\infty(\Omega)$, approximating it in L^2-norm by functions as above with supports contained in a common compact set in Ω, we see that (3.11) also holds for it. Now define R and R' as follows:

$$(3.12) \quad Rf(x,t)=\frac{\sigma(t)}{2\pi}\iint \exp\left(i\xi\int_{t'}^t \sigma(s)ds\right)(\chi(\xi)-1)f(x,t')dt'd\xi$$

$$(3.13) \quad R'f(x,t)=\frac{1}{2\pi}\iint \exp\left(i\xi\int_{t'}^t \sigma(s)ds\right)(\chi(\xi)-1)f(x,t')\sigma(t')dt'd\xi.$$

Then, (3.4) holds for $f\in C_0^\infty(\Omega)$. (3.5) can be proved in a similar way. Finally, inequalities in (3.7) follow easily from definitions (3.12) and (3.13). Q.E.D.

§4. L^2-estimate. Lemma 4.1. *Let E_j be the parametrix constructed in Theorem* 3.1. *If $f\in H_{0,0}^0(\Omega)$ and $(\phi\partial/\partial x)_P f\in H_{0,0}^0(\Omega)$ $(0\leqq p\leqq j)$ we can write*

$$(4.1) \quad \begin{aligned} \partial^p/\partial t^p E_j f&=\sum_{1\leqq k\leqq p}\sigma_{p,k}(t)E_j((\phi\partial/\partial x)^k f)\\ &+\sum_{0\leqq l+m\leqq p-1}\sigma_{p,l,m}(t)\partial^l/\partial t^l(\phi\partial/\partial x)^m(f+R_j f) \end{aligned}$$

where $\sigma_{p,k}$, $\sigma_{p,l,m}\in C^\infty((\alpha,\beta))$ are appropriate functions independent of f.

Proof. This can be proved by induction on p using Lemma 2.1 (2.3) and Theorem 3.1 (3.4). Q.E.D.

Lemma 4.2. *Let E_j be as in the above lemma. Choose any functions $\omega,\tilde{\omega}\in C_0^\infty(\Omega)$ such that $\tilde{\omega}=0$ near* supp ω, *and fix any integer p such that $0\leqq p\leqq j$. Then, $\omega E_j(\tilde{\omega}f)\in H_{p,q}$ for any positive integer q if $f\in H_{p,0}^{\mathrm{loc}}(\Omega)$, and we have with a constant C independent of f*

$$(4.2) \quad \|\omega E_j(\tilde{\omega}f)\|_{p,q}\leqq\|\tilde{\omega}f\|_{p,0} \qquad f\in H_{p,0}^{\mathrm{loc}}(\Omega).$$

Proof is omitted.

Theorem 4.3. *Let I,J be non negative integers. Assume that $u,(\phi\partial/\partial x)^k(Lu)$, and $\partial^l/\partial t^l(\phi\partial/\partial x)^m(Lu)\in H_{I,0}^{\mathrm{loc}}(\Omega)$ for $0\leqq k\leqq J$ and $0\leqq l+m\leqq J-1$, then $u\in H_{I,J}^{\mathrm{loc}}(\Omega)$. Take any two functions $\omega,\tilde{\omega}\in C_0^\infty(\Omega)$ such that $\tilde{\omega}=1$ near* supp ω, *and let l_ω be the number defined in Theorem* 3.1 (3.6), *then, for every positive integer N and $0<\delta<\frac{1}{2}(l_w+1)^{-1}$, we have with a constant C independent of u*

$$\|\omega u\|_{I,J} \leqq C\Big(\sum_{0\leqq k\leqq J} \|(\phi\partial/\partial x)^k(\tilde{\omega}f)\|_{I,-\delta} + \sum_{0\leqq l+m\leqq J-1} \|(\phi\partial/\partial x)^m(\tilde{\omega}f)\|_{I,l} + \|(L\tilde{\omega})u\|_{I,0} + \|\tilde{\omega}u\|_{I,-N}\Big) \tag{4.3}$$

where $f=Lu$.

Proof. Using Theorem 3.1 (3.5) with $j=I$, we can write

$$\omega u = \omega E_I(\tilde{\omega}f) + \omega E_I((L\tilde{\omega})u) - \omega R'_I(\tilde{\omega}u). \tag{4.4}$$

Hence (4.3) follows from Theorem 3.1 (3.6), (3.7) and Lemmas 4.1, 4.2.

Q.E.D.

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