39. A Classification of Compact 3-Manifolds with Infinite Cyclic Fundamental Groups

By Akio KAWAUCHI
Department of Mathematics, Kobe University

(Comm. by Kunihiko KODAIRA, M. J. A., March 12, 1974)

I. Results. We consider a compact connected piecewise linear 3-manifold M^3 which may be either orientable or non-orientable. If there is a component of the boundary ∂M^3 of M^3 which is homeomorphic to S^2, we attach a 3-cell to eliminate it. Note that the orientability of the resulting manifold coincides with that of the original one. Thus we assume that the boundary ∂M^3 contains no components which are homeomorphic to S^2 throughout this note. Under this assumption compact 3-manifolds with $\pi_1=Z$, Z being an infinite cyclic group will be classified modulo Poincaré Conjecture. The classification implies that such a manifold is essentially the S^2-bundle over S^1: $S^1\times S^2$, the twist S^2-bundle over S^1: $S^1\times_\tau S^2$, the solid torus: $S^1\times B^2$ or the solid Klein bottle: $S^1\times_\tau B^2$.

First, by using results of H. Kneser [2], J. H. C. Whitehead [8] and J. W. Milnor [3], we shall prove the following:

Theorem 1. *If $\partial M^3=\phi$ and $\pi_1(M^3)=Z$ then M^3 is homeomorphic to the connected sum $(S^1\times S^2)\#\tilde{S}^3$ or $(S^1\times_\tau S^2)\#\tilde{S}^3$ according as M^3 is orientable or non-orientable, where $\tilde{S}^3$ is a homotopy 3-sphere.*

Next, using Partial Poincaré Duality due to the present author [1], we shall obtain the following:

Theorem 2. *If $\partial M^3\neq\phi$ and $\pi_1(M^3)=Z$ then M^3 is homeomorphic to $(S^1\times B^2)\#\tilde{S}^3$ or $(S^1\times_\tau B^2)\#\tilde{S}^3$ according as M^3 is orientable or non-orientable. In particular, in case M^3 is orientable, M^3 may be considered as $cl(\tilde{S}^3$-unknotted solid torus).*

From Theorems 1 and 2 we obtain the following Conclusion:

Conclusion. *Any compact connected 3-manifold with $\pi_1=Z$ is homeomorphic to $(S^1\times S^2)\#\tilde{S}^3$, $(S^1\times_\tau S^2)\#\tilde{S}^3$, $(S^1\times B^2)\#\tilde{S}^3$ or $(S^1\times_\tau B^2)\#\tilde{S}^3$ with a finite number of open 3-cells removed.*

II. Sketch of proofs. Proofs will be considered in the piecewise linear category.

Proof of Theorem 1. By a result of H. Kneser [2], M^3 is homeomorphic to $P\#\tilde{S}^3$, where P is a prime 3-manifold in the sense that if P is homeomorphic to $P_1\#P_2$ then P_1 or P_2 is a 3-sphere. Since $\pi_1(P)=Z$, from the sphere theorem in the sense of J. H. C. Whitehead [8], we obtain a 2-sphere Σ in P which does not bound a 3-cell. Because P is prime Σ does not separate P. Hence cutting along Σ and attaching two 3-cells to eliminate the resulting boundaries, we have a closed manifold P'. Choose a 3-cell B^3 in P' containing these pasted two 3-cells in the interior. Then $P'=cl(P'-B^3)\cup B^3$ and the original manifold P becomes $cl(P'-B^3)\cup X$, where X is obtained from B^3 by removing disjoint two open 3-cells and then matching the resulting boundaries. Therefore P is homeomorphic to $P'\#(S^1\times S^2)$ or $P'\#(S^1\times_\tau S^2)$. Since P is prime, P is homeomorphic to $S^1\times S^2$ or $S^1\times_\tau S^2$. (The above technique appears in J. W. Milnor's paper [3, Lemma 1].) This completes the outlined proof.

Proof of Theorem 2. Let $p:\tilde{M}^3\to M^3$ be the universal covering which is obviously infinite cyclic. To prove that the homology $H_*(\tilde{M}^3;Z)$ is finitely generated, we need Lemma:

Lemma. *For each component F of ∂M^3, the canonical homomorphism $\pi_1(F)\to\pi_1(M^3)$ is non-trivial.*

Using this Lemma, each component of the preimage $p^{-1}(F)$ is an infinite cyclic covering space over F because $\pi_1(M^3)=Z$. (See [1, Lemma 3.1].) Hence each component of $\partial\tilde{M}^3$ is non-compact. This implies $H_2(\partial\tilde{M}^3;Z)=0$. By the Partial Poincaré Duality [1, Theorem 2.1], we have $H_2(\tilde{M}^3,\partial\tilde{M}^3;Z)\approx H^0(\tilde{M}^3;Z)\approx Z$. Using the homology exact sequence of the pair $(\tilde{M}^3,\partial\tilde{M}^3)$, we obtain that $H_2(\tilde{M}^3;Z)$ is finitely generated. Thus $H_*(\tilde{M}^3;Z)$ is finitely generated.

Again, applying the Partial Poincaré Duality, $H^i(\tilde{M}^3;Z)\approx H_{2-i}(\tilde{M}^3,\partial\tilde{M}^3;Z)$ for any i. An easy computation shows that $\tilde{M}^3$ is contractible. Hence M^3 is homotopy equivalent to S^1 and ∂M^3 is homeomorphic to the torus or the Klein bottle (Note that ∂M^3 is connected and Euler characteristic $\chi(\partial M^3)$ is equal to $2\chi(M^3)=2\chi(S^1)=0$). By the loop theorem [6], we can choose a proper 2-cell D in M^3 so that ∂D does not separate ∂M^3. Cutting along D, we obtain a manifold M^* whose boundary ∂M^* is a 2-sphere, for an easy computation implies $\chi(\partial M^*)=2$. Choose a 3-cell B^3 containing two copies of D so that $\Delta=cl(M^*-B^3)\cap B^3$ is a proper 2-cell in M^*. Then $M^*=cl(M^*-B^3)\cup B^3$ and the original manifold M^3 is homeomorphic to the disk sum $cl(M^*-B^3)\natural X$, where X is obtained from B^3 by matching disjoint two 2-cells in $\partial B^3-\Delta$. Therefore M^3 is homeomorphic to $cl(M^*-B^3)\natural(S^1\times B^2)$ or $cl(M^*-B^3)\natural(S^1\times_\tau B^2)$. Using $\pi_1(M^3)=Z$, we see that $cl(M^*-B^3)$ is a homotopy 3-cell. Hence M^3 is homeomorphic to $\tilde{S}^3\#(S^1\times B^2)$ or $\tilde{S}^3\#(S^1\times_\tau B^2)$. This completes the proof.

Proof of lemma. Suppose that for some component F of ∂M^3 the canonical homomorphism $\pi_1(F)\to\pi_1(M^3)$ is trivial. Since F is not a 2-sphere we can choose two simple polygonal loops l_1 and l_2 in F which intersect transversally in a single point. l_1, l_2 are null homotopic in M^3. Hence, by Dehn's lemma [5] or the loop theorem [6], there exist (polyhedral) 2-cells D_1, D_2 in M^3 bounded by l_1, l_2 such that $D_1 \cap \partial M^3 = l_1$, $D_2 \cap \partial M^3 = l_2$, respectively. Consider the intersection $D_1 \cap D_2$ and keep an eye on the intersection curve L starting from the intersection point $l_1 \cap l_2$. Then L must be an endless line. This is obviously impossible. Thus we prove Lemma.

III. Supplementary remarks. Theorem 2 can be also shown by using the Stallings fibration theorem [7] instead of the Partial Poincaré Duality. This duality for 3-manifolds is weaker than the Stallings fibration theorem, but more general. The following discussion shows the relation between them: If $\tilde{M}^3$ is a covering space associated with epimorphism $\gamma : \pi_1(M^3) \to Z$ and if $H_1(\tilde{M}^3; R)$ is finitely generated over a principal ideal domain R and if $\tilde{M}^3$ is orientable over R, the Partial Poincaré Duality implies that $H_*(\tilde{M}^3; R)$ is finitely generated and $H^i(\tilde{M}^3; G) \approx H_{2-i}(\tilde{M}^3, \partial\tilde{M}^3; G)$ for any i and any R-module G.

If ker $[\gamma : \pi_1(M^3) \to Z]$ is finitely generated and is not Z_2 and if M^3 is irreducible, Stallings showed that M^3 is a fiber bundle over S^1 whose fiber is a (proper) connected surface F in M^3. This implies that, in fact, $\tilde{M}^3$ splits: $(\tilde{M}^3, \partial\tilde{M}^3) = (F, \partial F) \times R^1$, hence, in case $\tilde{M}^3$ is orientable, there is a duality $H^i(\tilde{M}^3; Z) \approx {}_{2-i}(\tilde{M}^3, \partial\tilde{M}^3; Z)$ for any i.

On the other hand, if $H_1(M^3; Z) = Z$, then it can be shown that $H_*(\tilde{M}^3; Q)$ is finitely generated over Q. In case $\partial M^3 \neq \phi$, we see that ∂M^3 is homeomorphic to $S^1 \times S^1$ or $S^1 \times_\tau S^1$ according as M^3 is orientable or non-orientable. If $\tilde{M}^3$ is orientable, there is a duality $H^i(\tilde{M}^3; Q) \approx H_{2-i}(\tilde{M}^3, \partial\tilde{M}^3; Q)$ for any i. For $i=0$, we infer that $H^0(\tilde{M}^3; Z) \approx H_2(\tilde{M}^3, \partial\tilde{M}^3; Z) \approx Z$. The knot theory is known to be a non-trivial example of useful applications of this duality. (See J. W. Milnor [4].)

Because of the absence of the theory corresponding to the Stallings fibration theorem, the Partial Poincaré Duality is expected to be useful for 4-manifolds. For example, using this, the following is shown: *A locally unknotted 2-knot S^2 in 4-sphere S^4 (in the piecewise linear category) is algebraically unknotted if $\pi_1(S^4 - S^2) = Z$.* See [1].

References

[1] A. Kawauchi: On partial Poincaré duality and higher dimensional knots with $\pi_1 = Z$. Master Thesis, Kobe Univ. (1974).

[2] H. Kneser: Geschlossen Flächen in dreidimensionalen Mannigfaltigkeiten. Jber. Deutsch. Math. Verein., **38**, 248–260 (1929).

[3] J. W. Milnor: A unique decomposition theorem for 3-manifolds. Amer. J. Math., **84**, 1–7 (1962).

[4] ——: Infinite cyclic coverings. Conference on the Topology of Manifold (Michigan State Univ. E. Lansing, Michigan 1967). Prindle, Weber and Schmidt, Boston, Mass. (1968).

[5] C. D. Papakyriakopoulos: On Dehn's lemma and the asphericity of knots. Ann. of Math., **66**, 1–26 (1957).
[6] J. Stallings: On the loop theorem. Ann. of Math., **72**, 12–19 (1960).
[7] ——: On Fibering Certain 3-Manifolds. Topology of 3-Manifolds and Related Topics (ed. M. K. Fort, Jr.). Prentice Hall, pp. 110–117 (1962).
[8] J. H. C. Whitehead: On finite cocycles and the sphere theorem. Colloq. Math., **6**, 271–281 (1957).