132. *On Sylow Subgroups and an Extension of Groups*

By Zensiro GOSEKI
Gunma University

(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

Let A and B be groups. If there are homomorphisms f and g such that a sequence $\xrightarrow{f} A \xrightarrow{g} B \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{f}$ is exact, then we denote this collection by $(A, B: f, g)$ and we say $(A, B: f, g)$ to be *well defined*. Let $(A, B: f, g)$ and $(C, D: f_1, g_1)$ be well defined. If C and D are subgroups of A and B, respectively, and if $f=f_1$ on C and $g=g_1$ on D, then we call $(C, D: f_1, g_1)$ a *subgroup* of $(A, B: f, g)$ and in this case, we denote $(C, D: f_1, g_1)$ by $(C, D: f, g)$. Furthermore, we call $(C, D: f, g)$ a *normal subgroup* of $(A, B: f, g)$ if $C \lhd A$ and $D \lhd B$, and a *Sylow subgroup* of $(A, B: f, g)$ if C is a Sylow subgroup of A (in this case D is also a Sylow subgroup of B). We shall discuss the existence of such Sylow subgroups $(C, D: f, g)$ of $(A, B: f, g)$. It is easy to see that there are homomorphisms f and g such that $(A, B: f, g)$ is well defined iff there are groups M, N and homomorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that sequences $1 \to M \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} N \to 1$ and $1 \to N \xrightarrow{\beta_1} B \xrightarrow{\beta_2} M \to 1$ are exact. This shows that the results given in this note are related to an extension of groups.

Lemma 1. *Let $(A, B: f, g)$ be well defined. Let M and N be subgroups of A and B, respectively. Then $(M, N: f, g)$ is well defined iff $f(M)=f(A)\cap N$ and $g(N)=g(B)\cap M$.*

Proof. Since $(A, B: f, g)$ is well defined, $A/g(B)\cong f(A)$ and so $M/M\cap g(B)\cong Mg(B)/g(B)\cong f(M)$. Assume that $(M, N: f, g)$ is well defined. Then $M/g(N)\cong f(M)$. Hence $M/g(N)\cong M/M\cap g(B)$ where this isomorphism is given by $xg(N)\to x(M\cap g(B))$ for all $x\in M$. Hence $M\cap g(B)=g(N)$. Similarly $N\cap f(A)=f(M)$. Conversely, let $f(M)=N\cap f(A)$ and $g(N)=M\cap g(B)$. Then $M/g(N)=M/M\cap g(B)\cong Mg(B)/g(B)\cong f(M)$, i.e., $M/g(N)\cong f(M)$ where this isomorphism is given by $xg(N)\to f(x)$ for all $x\in M$. Similarly $N/f(M)\cong g(N)$ where this isomorphism is given by $yf(M)\to g(y)$ for all $y\in N$. Hence $(M, N: f, g)$ is well defined.

Lemma 2. *Let $(A, B: f, g)$ be well defined and let $(M, N: f, g)$ be a normal subgroup of $(A, B: f, g)$. Then $(A/M, B/N: \bar{f}, \bar{g})$ is well defined, where $\bar{f}$ and $\bar{g}$ are homomorphisms which are naturally induced by f and g, respectively.*

Proof. *By Lemma* 1, $f(A)\cap N=f(M)$. *Hence* $f^{-1}(N)=f^{-1}(f(A)$

$\cap N)=f^{-1}(f(M))=Mg(B)$. Similarly $g^{-1}(M)=Nf(A)$. Hence a sequence $\xrightarrow{\bar{g}} A/M \xrightarrow{\bar{f}} B/N \xrightarrow{\bar{g}} A/M \xrightarrow{\bar{f}} B/N \xrightarrow{\bar{g}}$ is exact.

Lemma 3. *Let M and N be subgroups of groups A and B, respectively, and let $P \lhd M$ and $Q \lhd N$. If $(A, B: f, g)$, $(M/P, N/Q: \bar{f}, \bar{g})$ and $(P, Q: f, g)$ are well defined where $\bar{f}$ and $\bar{g}$ are homomorphisms which are naturally induced by f and g, respectively, then $(M, N: f, g)$ is well defined.*

Proof. Since $(P, Q: f, g)$ is well defined, $f(A)\cap Q=f(P)$ and $g(B)\cap P=g(Q)$. Hence $f(M)\cap Q=f(M)\cap f(A)\cap Q=f(M)\cap f(P)=f(P)$. Similarly $g(N)\cap P=g(Q)$. Since $(M/P, N/Q: \bar{f}, \bar{g})$ is well defined, $N/Q/f(M)Q/Q \cong g(N)P/P$. Hence $N/f(M)Q \cong N/Q/f(M)Q/Q \cong g(N)P/P \cong g(N)/g(N)\cap P=g(N)/g(Q)$, i.e., $N/f(M)Q\cong g(N)/g(Q)$ and hence $f(M)Q$ is a kernel of a homomorphism $N\to g(N)/g(Q)$ given by $x\to g(x)g(Q)$ for all $x\in N$. On the other hand, for any $x\in N$, $g(x)\in g(Q)$ iff $x\in(N\cap f(A))Q$ since a kernel of the homomorphism $g: B\to A$ is $f(A)$. Hence $f(M)Q=(N\cap f(A))Q$. Hence $f(M)/f(P)=f(M)/f(M)\cap Q\cong f(M)Q/Q=(N\cap f(A))Q/Q\cong N\cap f(A)/N\cap f(A)\cap Q=N\cap f(A)/f(P)$, i.e., $f(M)/f(P)\cong N\cap f(A)/f(P)$ where this isomorphism is given by $f(x)f(P)\to f(x)f(P)$ for all $x\in M$. Thus $f(M)=N\cap f(A)$. Similarly $g(N)=M\cap g(B)$. By Lemma 1, $(M, N: f, g)$ is well defined.

In the rest of this note we consider only the finite groups. For a prime number p, a following result is well known (see [2, Lemma 2.1]).

Lemma 4. *Let $N \lhd A$ and S_p a Sylow p-subgroup of A. Then $N\cap S_p$ and NS_p/N are the Sylow p-subgroups of N and A/N, respectively.*

Lemma 5. *Let $(A, B: f, g)$ be well defined. Let S_p be a Sylow p-subgroup of A and T a subgroup of B. Then $(S_p, T: f, g)$ is well defined iff T is a Sylow p-subgroup of $g^{-1}(S_p)$ and $f(S_p)\subseteq T$. In this case, T is a Sylow p-subgroup of B.*

Proof. Since $(A, B: f, g)$ is well defined, $|A|=|B|$. Assume that $(S_p, T: f, g)$ is well defined. Then $|S_p|=|T|$. Hence T is a Sylow p-subgroup of B and so of $g^{-1}(S_p)$. Clearly $f(S_p)\subseteq T$. Conversely, let $f(S_p)\subseteq T$ and let T be a Sylow p-subgroup of $g^{-1}(S_p)$. By Lemma 4, $S_pg(B)/g(B)$ is a Sylow p-subgroup of $A/g(B)$. On the other hand, $A/g(B)\cong f(A)$ and this induces an isomorphism $S_pg(B)/g(B)\cong f(S_p)$. Hence $f(S_p)$ is a Sylow p-subgroup of $f(A)$. Furthermore, since the isomorphism $B/f(A)\cong g(B)$ is given by $bf(A)\to g(b)$ for all $b\in B$, $g^{-1}(S_p)/f(A)\cong g(B)\cap S_p$. Therefore, if $|g(B)\cap S_p|=p^n$ and $|f(S_p)|=p^m$, then $|g^{-1}(S_p)|=p^{n+m}q$ where q is an integer such that $p\nmid q$. Hence $|T|=p^{n+m}$. Thus $p^n=|T/f(S_p)|=|g(B)\cap S_p|=|g^{-1}(S_p)/f(A)|$. Since $T\cap f(A)$ and $f(S_p)$ are Sylow p-subgroups of $f(A)$ and since $f(S_p)\subseteq T\cap f(A)$, we have $T\cap f(A)=f(S_p)$. Thus $T/f(S_p)$ is naturally embedded into $g^{-1}(S_p)/f(A)$ and so embedded into $g(B)\cap S_p$. Since those have the same order, $T/f(S_p)\cong g(B)\cap S_p$ and this isomorphism is given by $tf(S_p)\to g(t)$ for all $t\in T$. Consequently $g(T)=g(B)\cap S_p$. Therefore, by Lemma 1, $(S_p, T: f, g)$ is well defined.

By the above result, if $(A, B: f, g)$ is well defined and if S_p is a Sylow p-subgroup of A then there is a Sylow p-subgroup T_p of B such that $(S_p, T_p: f, g)$ is well defined. We denote by $n_p(A)$ and $n_p(A, B: f, g)$ the number of the Sylow p-subgroups of A and $(A, B: f, g)$, respectively.

Theorem 1. *Let $(A, B: f, g)$ be well defined. Then*:

(1) *If S_p is a Sylow p-subgroup of A, then the number t of Sylow p-subgroups T_p of B such that $(S_p, T_p: f, g)$ is well defined is independent of a choice of S_p and $t\equiv 1 \bmod p$.*

(2) $n_p(A, B: f, g)\equiv 1 \bmod p$.

(3) $n_p(A, B: f, g)=n_p(A)n_p(B)/n_p(f(A))n_p(g(B))$.

Proof. (1) By Lemma 5, t is the number of Sylow p-subgroups of $g^{-1}(S_p)$ which contain $f(S_p)$. Hence $t\equiv 1 \bmod p$ (see [1, p. 152]). We shall prove that t is independent of a choice of S_p. Let T_p be a subgroup of B such that $(S_p, T_p: f, g)$ is well defined. Then $g(B)\cap S_p=g(T_p)$. Hence $g^{-1}(S_p)=T_pf(A)$. Now let $\mathfrak{T}$ be a set of Sylow p-subgroups of $T_pf(A)$ and $\mathfrak{S}$ a set of Sylow p-subgroups of $f(A)$. Let $\mu: \mathfrak{T}\to\mathfrak{S}$ be a map defined by $\mu(T)=T\cap f(A)$ for all $T\in\mathfrak{T}$. Then μ is well defined. Let $J\in\mathfrak{S}$. There is $a\in A$ such that $J=f(a)^{-1}(T_p\cap f(A))f(a)$. Since $g(f(a)^{-1}T_pf(a))=g(T_p)\subseteq S_p$, $f(a)^{-1}T_pf(a)\in\mathfrak{T}$ and $\mu(f(a)^{-1}T_pf(a))=J$. Consequently μ is surjective. Next let $J_1, J_2\in\mathfrak{S}$. Then there is $a\in A$ such that $f(a)^{-1}J_1f(a)=J_2$. Next let $\delta: \mu^{-1}(J_1)\to\mu^{-1}(J_2)$ be a map defined by $\delta(T)=f(a)^{-1}Tf(a)$ for all $T\in\mu^{-1}(J_1)$. Then δ is well defined and bijective. On the other hand, $f(S_p)$ is a Sylow p-subgroup of $f(A)$ and so $f(S_p)\in\mathfrak{S}$. Moreover, $\mu^{-1}(f(S_p))$ is a set of Sylow p-subgroups of $T_pf(A)$ which contain $f(S_p)$. Hence $t=n_p(T_pf(A))/n_p(f(A))$. Generally $n_p(Tf(A))=n_p(T'f(A))$ for any Sylow p-subgroups T and T' of B because $Tf(A)$ and $T'f(A)$ are isomorphic. Hence t is independent of a choice of a Sylow p-subgroup S_p of A.

(2) By (1), $n_p(A, B: f, g)=n_p(A)t$. Since $n_p(A)\equiv t\equiv 1 \bmod p$, (2) holds.

(3) From the proof of (1) stated above,

$$n_p(A, B: f, g)=n_p(A)n_p(T_pf(A))/n_p(f(A))$$

where $(S_p, T_p: f, g)$ is well defined and a Sylow p-subgroup of $(A, B: f, g)$. By [2, Theorem 2.1],

$$n_p(A)=n_p(g(B))n_p(f(A))n_p(N_{S_pg(B)}(g(T_p))/g(T_p)),$$
$$n_p(B)=n_p(f(A))n_p(g(B))n_p(N_{T_pf(A)}(f(S_p))/f(S_p))$$

and

$$n_p(T_pf(A))/n_p(f(A))=n_p(N_{T_pf(A)}(f(S_p))/f(S_p)).$$

Hence $n_p(A,B:f,g)=n_p(A)n_p(B)/n_p(f(A))n_p(g(B))$.

Theorem 2. *Let $(A,B:f,g)$ be well defined and let $(P,Q:f,g)$ be a subgroup of $(A,B:f,g)$. If P is a p-subgroup of A (hence Q is also a p-subgroup of B), then there is a Sylow p-subgroup $(S_p,T_p:f,g)$ of $(A,B:f,g)$ such that $(P,Q:f,g)$ is a subgroup of $(S_p,T_p:f,g)$.*

Proof. Let T be a Sylow p-subgroup of the group $f(A)Q$ such that $Q\subseteq T$. Let $M=f(A)\cap T$. By Lemma 4, M is a Sylow p-subgroup of $f(A)$. If S is a Sylow p-subgroup of A, then $f(S)$ is a Sylow p-subgroup of $f(A)$ and so there is $a\in A$ such that $f(a)^{-1}f(S)f(a)=M$. Hence $a^{-1}Sa\subseteq f^{-1}(M)$. This shows that any Sylow p-subgroup of $f^{-1}(T)$ is a Sylow p-subgroup of A. Let S_p be a Sylow p-subgroup of $f^{-1}(T)$ such that $P\subseteq S_p$. Then S_p is a Sylow p-subgroup of A. Since $g(T)\subseteq g(f(A)Q)=g(Q)\subseteq P\subseteq S_p$, $T\subseteq g^{-1}(S_p)$. Now let T_p be a Sylow p-subgroup of $g^{-1}(S_p)$ such that $T\subseteq T_p$. Then $f(S_p)\subseteq T_p$ since $f(S_p)\subseteq T$. Hence, by Lemma 5, $(S_p,T_p:f,g)$ is well defined. Furthermore $Q\subseteq T_p$ and $P\subseteq S_p$. This completes our proof.

Theorem 3. *Let $(A,B:f,g)$ be well defined, let $(K,L:f,g)$ be a normal subgroup of $(A,B:f,g)$ and $(S_p,T_p:f,g)$ a Sylow p-subgroup of $(A,B:f,g)$. Then $(K\cap S_p,L\cap T_p:f,g)$, $(S_pK/K,T_pL/L:\bar{f},\bar{g})$ and $(S_pK,T_pL:f,g)$ are well defined where $\bar{f}$ and $\bar{g}$ are homomorphisms which are naturally induced by f and g, respectively.*

Proof. $K\cap S_p$ is a Sylow p-subgroup of K. Since $L\cap T_p$ is a Sylow p-subgroup of L, $L\cap T_p$ is also a Sylow p-subgroup of $g^{-1}(K\cap S_p)\cap L$. Furthermore $f(K\cap S_p)\subseteq L\cap T_p$. Hence, by Lemma 5, $(K\cap S_p, L\cap T_p:f,g)$ is well defined. By Lemma 4, S_pK/K and T_pL/L are Sylow p-subgroups of A/K and B/L, respectively. Furthermore, by Lemma 2, $(A/K,B/L:\bar{f},\bar{g})$ is well defined and $\bar{f}(S_pK/K)\subseteq T_pL/L\subseteq\bar{g}^{-1}(S_pK/K)$. Hence, by Lemma 5, $(S_pK/K,T_pL/L:\bar{f},\bar{g})$ is well defined. Therefore, by Lemma 3, $(S_pK,T_pL:f,g)$ is well defined.

References

[1] W. Burnside: Theory of Groups of Finite Order (2nd ed.). Dover, New York. MR 16, 1086 (1955).

[2] M. Hall: On the number of Sylow subgroups in a finite group. J. Algebra, **7**, 363–371 (1967).