162. The Semi-discretisation Method and Nonlinear Time-dependent Parabolic Variational Inequalities

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1. Introduction. Let H be a (real) Hilbert space and X be a reflexive Banach space such that $X\subset H$, X is dense in H and the natural injection from X into H is continuous. We denote by X^* the dual space of X. Identifying H with its dual, we have the relations: $X\subset H\subset X^*$. Throughout this paper, let $0<T<\infty$, $1<p<\infty$ and $1/p+1/p'=1$. Let $K=\{K(t); 0\le t\le T\}$ be a family of closed convex subsets of X, ψ be a function on $[0,T]\times X$ such that for each $t\in[0,T]$, $\psi(t;\cdot)$ is a lower semicontinuous convex function on X with values in $(-\infty,\infty]$, and j be a continuous function on $[0,T]\times X$ such that for each $t\in[0,T]$, $j(t;\cdot)$ is convex on X. Suppose further that j is bounded on each bounded subset of $[0,T]\times X$ and for each $v\in L^p(0,T;X)$, $t\to\psi(t;v(t))$ is measurable. Then, for given $f\in L^{p'}(0,T;X^*)$ and $u_0\in X$ we mean by $V[K,j,\psi,f,u_0]$ the following problem: Find $u\in L^p(0,T;X)$ together with $u^*\in L^{p'}(0,T;X^*)$ such that

(i) u is an H-valued continuous function on $[0,T]$ with $u(0)=u_0$;

(ii) $u(t)\in K(t)$ for a.a. (almost all) $t\in(0,T)$ and $\psi(\cdot;u(\cdot))\in L^1(0,T)$;

(iii) $u^*(t)\in\partial j(t;u(t))$ for a.a. $t\in(0,T)$, where $\partial j(t;\cdot)$ is the subdifferential of $j(t;\cdot)$;

(iv) $u'=(\mathrm{d}/\mathrm{d}t)u\in L^2(0,T;H)$;

(v) $$\int_0^T(u'(t),u(t)-v(t))_H\mathrm{d}t+\int_0^T(u^*(t)-f(t),u(t)-v(t))_X\mathrm{d}t \le\int_0^T\{\psi(t;v(t))-\psi(t;u(t))\}\mathrm{d}t$$

for all $v\in L^p(0,T;X)\cap L^2(0,T;H)$ such that $v(t)\in K(t)$ for a.a. $t\in(0,T)$ and $\psi(\cdot;v(\cdot))\in L^1(0,T)$, where $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_H$ stand for the natural pairing between X^* and X and the inner product in H, respectively.

Remark. If we take $\psi(t;\cdot)+I_{K(t)}(\cdot)$ instead of $\psi(t;\cdot)$ we can formulate the above problem without using $K(t)$, where $I_{K(t)}$ is the indicator function of $K(t)$.

Many results on the existence, uniqueness and regularity of solutions of this kind of problems have been established by many authors (e.g., [1], [2], [4]-[6], [8]-[11]). Brézis [2] and Moreau [6] treated the case where $\psi(t;\cdot)$ is the indicator function of $K(t)$; in this case, the domain

$D(\partial\psi(t;\cdot))$ of $\partial\psi(t;\cdot)$ depends on t, since $K(t)=D(\partial\psi(t;\cdot))$. Watanabe [11] dealt with the case where $D(\partial\psi(t;\cdot))$ depends on t but $\{z;\psi(t;z)<\infty\}$ does not depend on t. Also, a result of Péralba [8] is interesting. Their method, except Moreau's one, is based on the nonlinear semigroup theory. In this paper, we shall show the existence of a solution of $V[K,j,\psi,f,u_0]$ by employing the semi-discretisation method with respect to t (cf. Raviart [9]).

2. Main results. Let us denote by $\mathcal{K}$ the set of all $v\in L^p(0,T;X)$ such that $v(t)\in K(t)$ for a.a. $t\in(0,T)$. We say that the mapping $t\to K(t)$ (resp. $t\to\psi(t;\cdot)$) is right continuous at $t=t_0$, if for any sequence $\{t_n\}$, $t_n\downarrow t_0$, the sequence $\{K(t_n)\}$ (resp. $\{\psi(t_n;\cdot)\}$) converges to $K(t_0)$ (resp. $\psi(t_0;\cdot)$) in X as $n\to\infty$ in the sense of Mosco [7]. Now, suppose that

(H. 1) *the mappings $t\to K(t)$ $t\to j(t;\cdot)$ and $t\to\psi(t;\cdot)$ are right continuous on* $[0,T]$;

(H. 2) *if t is a point in $[0,T]$ and z is an element of $K(t)$ with $\psi(t;z)<\infty$, then for each $s\in[t,T]$ there is $\tilde{z}\in K(s)$ such that*

$$\|\tilde{z}-z\|_X\le L|t-s|,$$
$$j(s;\tilde{z})-j(t;z)\le L|t-s|(1+\|z\|_X^p),$$
$$\psi(s;\tilde{z})-\psi(t;z)\le L|t-s|(1+\|z\|_X^p+|\psi(t;z)|);$$

(H. 3) $j(t;z)\ge C\|z\|_X^p-M$ *for all* $t\in[0,T]$ *and* $z\in K(t)$;

(H. 4) *there are an X^*-valued continuous function b_0 on $[0,T]$ and a real-valued continuous function b_1 on $[0,T]$ such that*

$$\psi(t;z)\ge(b_0(t),z)_X+b_1(t)\qquad \text{for all } t\in[0,T] \text{ and } z\in X;$$

(H. 5) *there is an X-valued Lipschitz continuous function h_0 on $[0,T]$ with L as a Lipschitz constant such that $h_0(t)\in K(t)$ for all $t\in[0,T]$ and $\psi(\cdot;h_0(\cdot))$ is bounded on $[0,T]$; where C,M,L are positive constants and $\|\cdot\|_X$ denotes the norm in X.*

Then we have

Theorem 1. *Let u_i $(i=1,2)$ be a solution of $V[K,j,\psi,f_i,u_{0,i}]$. Then the following holds:*

$$\|u_1(t_1)-u_2(t_1)\|_H^2-\|u_1(t_2)-u_2(t_2)\|_H^2\le\int_{t_2}^{t_1}2(f_1-f_2,u_1-u_2)_X dt$$

for any $t_1,t_2\in[0,T]$, $t_1\ge t_2$, where $\|\cdot\|_H$ denotes the norm in H.

Theorem 2. *Assume that $f,f'\in L^{p'}(0,T;X^*)$, $u_0\in K(0)$ and $\psi(0,u_0)<\infty$. Then there is a solution u of $V[K,j,\psi,f,u_0]$ such that $u\in L^\infty(0,T;X)$ (hence it is an X-valued weakly continuous function on $[0,T]$) and $\psi(\cdot;u(\cdot))\in L^\infty(0,T)$.*

Remark. If, for a solution u, property (iii) is not required, then assumptions on the t-dependence for mappings $t\to K(t)$, $t\to j(t;\cdot)$ and $t\to\psi(t;\cdot)$ may be weakened.

Remark. Theorem 2 has various applications to initial boundary value problems for nonlinear time-dependent parabolic differential equations (see [4]).

3. Sketch of proof of Theorems. Theorem 1 can be easily proved. We shall give below the outline of the proof of Theorem 2.

Let N be a positive integer and set $\varepsilon_N = T/N$ and

$$f_{N,n} = \varepsilon_N^{-1} \int_{\varepsilon_N(n-1)}^{\varepsilon_N n} f(t)\mathrm{d}t, \qquad n=1,2,\cdots,N.$$

Now, define a sequence $\{u_{N,n}, u_{N,n}^*\}_{n=1}^N \subset X \times X^*$ as follows: Let $u_{N,0} = u_0$ and $(u_{N,n}, u_{N,n}^*) \in X \times X^*$, $n=1,2,\cdots,N$, be a pair such that $u_{N,n} \in K(\varepsilon_N n)$, $u_{N,n}^* \in \partial j(\varepsilon_N n; u_{N,n})$ and the following holds:

$$\text{(3.1)} \quad \begin{cases} \varepsilon_N^{-1}(u_{N,n}-u_{N,n-1}, u_{N,n}-w)_H + (u_{N,n}^* - f_{N,n}, u_{N,n}-w)_X \\ \qquad \le \psi(\varepsilon_N n; w) - \psi(\varepsilon_N n; u_{N,n}) \qquad \text{for all } w \in K(\varepsilon_N n). \end{cases}$$

Note that by virtue of a result of Browder [3; Theorem 2], for $n=1,2,\cdots,N$, there exists such a pair $(u_{N,n}, u_{N,n}^*) \in X \times X^*$.

Substituting $h_0(\varepsilon_N n)$ for w in (3.1) and using hypotheses (H. 3), (H. 4) and (H. 5) we obtain

Lemma 1. *For some positive constant M_1 independent of N, n we have:*

$$\varepsilon_N \sum_{n=1}^N \|u_{N,n}\|_X^p \le M_1, \qquad \varepsilon_N \sum_{n=1}^N |\psi(\varepsilon_N n; u_{N,n})| \le M_1.$$

Next, by (H. 2), for each $n=1,2,\cdots,N$, there is $\tilde{u}_{N,n} \in K(\varepsilon_N n)$ such that

$$\|u_{N,n-1} - \tilde{u}_{N,n}\|_X \le \varepsilon_N L,$$
$$j(\varepsilon_N n; \tilde{u}_{N,n}) - j(\varepsilon_N(n-1); u_{N,n-1}) \le \varepsilon_N L(1+\|u_{N,n-1}\|_X^p),$$
$$\psi(\varepsilon_N n; \tilde{u}_{N,n}) - \psi(\varepsilon_N(n-1); u_{N,n-1})$$
$$\le \varepsilon_N L(1+\|u_{N,n-1}\|_X^p + |\psi(\varepsilon_N(n-1); u_{N,n-1})|).$$

Taking $\tilde{u}_{N,n}$ for w in (3.1) and making use of Lemma 1 we get

Lemma 2. *There is a positive constant M_2 independent of N, n such that*

$$\max_{1\le n\le N} \|u_{N,n}\|_X^p \le M_2, \qquad \max_{1\le n\le N} |\psi(\varepsilon_N n; u_{N,n})| \le M_2$$

and

$$\varepsilon_N^{-1} \sum_{k=1}^N \|u_{N,n} - u_{N,n-1}\|_H^2 \le M_2.$$

Remark. In case K, j and ψ are independent of t, the estimates in Lemmas 1 and 2 are due to Raviart [9].

We set $u_N(t) = u_{N,n}$, $u_N^*(t) = u_{N,n}^*$ and $\nabla_N u_N(t) = \varepsilon_N^{-1}(u_{N,n} - u_{N,n-1})$ if $t \in I_{N,n} = [\varepsilon_N(n-1), \varepsilon_N n)$, $n=1,2,\cdots,N$. As was seen above, sequences $\{u_N\}_{N\ge1}$, $\{u_N^*\}_{N\ge1}$ and $\{\nabla_N u_N\}_{N\ge1}$ are bounded in $L^\infty(0,T;X)$, $L^\infty(0,T;X^*)$ and $L^2(0,T;H)$, respectively. Therefore there are some subsequences $\{u_{N'}\}$, $\{u_{N'}^*\}$, $\{\nabla_{N'} u_{N'}\}$ such that $u_{N'} \to u$ weakly* in $L^\infty(0,T;X)$, $u_{N'}^* \to u^*$ weakly* in $L^\infty(0,T;X^*)$ and $\nabla_{N'} u_{N'} \to v$ weakly in $L^2(0,T;H)$ as $N' \to \infty$. Then we can show that $v = u'$ and from (H. 1) that $u \in \mathcal{K}$ and $\psi(\cdot\,; u(\cdot)) \in L^\infty(0,T)$. Also we see that u is an X-valued weakly continuous function on $[0,T]$ with $u(0) = u_0$ and $u^*(t) \in \partial j(t; u(t))$ for a.a. $t \in (0,T)$. Thus

u and u^* have properties (i)$\sim$(iv). Finally, by using the following lemma we can prove that (v) is fulfilled.

Lemma 3. *For each* $v \in \mathcal{K} \cap L^2(0, T; H)$ *such that* $\psi(\cdot\,; v(\cdot)) \in L^1(0, T)$, *there exists a sequence* $\{v_N\}$ *in* $L^p(0, T; X) \cap L^2(0, T; H)$ *such that* $v_N(t) \in K(\varepsilon_N n)$ *for* $t \in I_{N,n}$ $(n=1, 2, \cdots, N)$, $v_N \to v$ *strongly both in* $L^p(0, T; X)$ *and in* $L^2(0, T; H)$ *as* $N \to \infty$ *and* $\Phi_N(v_N) \to \Phi(v)$ *as* $N \to \infty$, *where*

$$\Phi(v) = \int_0^T \psi(t; v(t))\mathrm{d}t \quad \text{and} \quad \Phi_N(v_N) = \sum_{n=1}^{N} \int_{\varepsilon_N(n-1)}^{\varepsilon_N n} \psi(\varepsilon_N n; v_N(t))\mathrm{d}t.$$

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