77. Notes on Complex Lie Semigroups

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1. By a complex space, we mean a reduced, Hausdorff, complex analytic space. A semigroup S is called *a complex Lie semigroup* if and only if (1) S is a complex space and (2) the product $(x, y) \in S \times S \rightarrow xy \in S$ is a holomorphic map.

Important examples of complex Lie semigroups are: (1) a (finite dimensional, associative) C-algebra with respect to its product, (2) End (G) = the set of all (holomorphic) endomorphisms of a connected complex Lie group G (c.f., Chevalley [1]) and (3) Hol (V, V) = the set of all holomorphic maps of a compact complex space V into itself (Douady [2]).

The purpose of this note is to state some results on the structures of complex Lie semigroups with 1 (the identity) or 0 (zero). Details will be published elsewhere.

2. By *a subsemigroup* (resp. *an ideal*) *of a complex Lie semigroup*, we mean a subsemigroup (resp. an ideal) in the usual sense which is at the same time a closed complex subvariety. By *an isomorphism of complex Lie semigroups*, we mean an isomorphism as semigroups which is at the same time a biholomorphic map.

3. We first state the following Theorems 1 and 2.

Theorem 1. *Let S be a complex Lie semigroup with* 1 (*the identity*). *We denote by $G(S)$ the set of all invertible elements of S. Then* (1) *$G(S)$ is a non-singular open subspace of S and is a complex Lie group with respect to the product in S,* (2) *the closure $\overline{G(S)}$ is a union of some irreducible components of S and is a subsemigroup of S and* (3) *$\overline{G(S)} - G(S)$ is an ideal of $\overline{G(S)}$.*

Corollary. *Let V be a compact complex space. Then* Aut (V) (*the group of all biholomorphic maps of V onto itself*) *is* (*open and*) *closed in* Hol (V, V) *with respect to the compact-open topology.*

Theorem 2. *Let S be a complex Lie semigroup with* 1. *Assume that S is irreducible as a complex space. Then* (1) *the set of all singular points of an ideal of S is also an ideal of S,* (2) *each irreducible component of an ideal of S is also an ideal of S and* (3) *any ideal of S is written as a finite union of ideals of S which are irreducible as complex spaces.*

Now, let S be a complex Lie semigroup with 0 (zero). Locally, we extend the product map to a holomorphic map $M: \Omega' \times \Omega' \to \Omega$, where Ω' and $\Omega(\Omega' \subset \Omega)$ are ambient spaces of open neighbourhoods of 0 in S. We assume that $\dim \Omega' = \dim \Omega = \dim T_0S$, where T_0S is the (Zariski) tangent space to S at 0 (see, e.g., [3]). Expressing the map M in a coordinate system in Ω, we expand it into the power series at $(0,0)$ as follows:

$$M^k(x^1, \cdots, x^n, y^1, \cdots, y^n) = \sum_{i,j} (\partial^2 M^k / \partial x^i \partial y^j)_{(0,0)} x^i y^j + (\text{higher order terms}),$$

$k = 1, \cdots, n$ ($n = \dim T_0S$). We define a product in T_0S as follows:

$$XY = \sum_{i,j,k} (\partial^2 M^k / \partial x^i \partial y^j)_{(0,0)} u^i v^j (\partial / \partial z^k)_0,$$

where $X = \sum_i u^i (\partial / \partial z^i)_0$ and $Y = \sum_j v^j (\partial / \partial z^j)_0$. Then

Theorem 3. *Let S be a complex Lie semigroup with 0. Then the (Zariski) tangent space T_0S to S at 0 has a structure of an (associative) C-algebra.*

As for the complex space structures of complex Lie semigroups, we have the following theorem, which is an easy consequence of M. Kato's theorem [4].

Theorem 4. *Let S be a complex Lie semigroup with 1 and 0 such that $\overline{G(S)} = S$. Then (1) S is holomorphically imbedded in a complex number space C^N as an affine subvariety and (2) if 0 is a non-singular point of S, then S is biholomorphic to C^n ($n = \dim S$).*

Thus, by Theorem 4, a connected, non-singular, complex Lie semigroup with 1 and 0 is biholomorphic to C^n. ($\overline{G(S)} = S$ by (2) of Theorem 1). However, it may not in general be isomorphic to any C-algebra (as complex Lie semigroups). A criterion is

Theorem 5. *Let S be a connected, non-singular, complex Lie semigroup with 1 and 0. Then S is isomorphic to a C-algebra (as complex Lie semigroups) if and only if the C-algebra T_0S has the identity. In fact, if this is the case, then S is isomorphic to T_0S.*

Example. We put $S = C^2$ and define a product in S as follows: for (a, b) and $(c, d) \in S$,

$$(a, b)(c, d) = (ac, a^2 d + bc^2).$$

Then S is a connected, non-singular, complex Lie semigroup with the identity $(1, 0)$ and zero $(0, 0)$. But the C-algebra $T_{(0,0)}S$ does not have the identity. Thus S is not isomorphic to any C-algebra.

References

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