117. *Factorization of a Hyponormal Operator*

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1. In this paper, only bounded linear operators on a fixed Hilbert space H will be considered. An operator T is said to be hyponormal if

$$T^*T-TT^*\geqq 0.$$

This note is motivated by a recent work [1] of Yoshino, and we prove the following theorem.

Theorem. *Let A, B and S be operators such that*

(i) $B\geqq A\geqq 0$,

(ii) $\|S\|\leqq 1$,

(iii) $S^*AS=B$.

Then the operator $T=A^{1/2}S$ is a hyponormal operator.

Conversely, if T is a hyponormal operator, then there exist operators A, B and S which satisfy (i), (ii) and (iii), and T can be written in the form $T=A^{1/2}S$.

2. Proof of the Theorem. Suppose that there exist operators A, B and S which satisfy (i), (ii) and (iii). Then

$$\begin{aligned}(A^{1/2}S)^*(A^{1/2}S)-(A^{1/2}S)(A^{1/2}S)^* &= S^*AS-A^{1/2}SS^*A^{1/2}\\ &= B-A^{1/2}SS^*A^{1/2}\geqq A-A^{1/2}SS^*A^{1/2} \qquad (1)\\ &= A^{1/2}(I-SS^*)A^{1/2}\geqq 0.\end{aligned}$$

Conversely, suppose that T is a hyponormal operator. Let

$$T^*=U(TT^*)^{1/2}$$

be a polar decomposition of T^*. Let $A=TT^*$ and $B=T^*T$. Then, since T is hyponormal we have $B\geqq A\geqq 0$. Also, we have

$$B=T^*T=U(TT^*)^{1/2}(TT^*)^{1/2}U^*=UTT^*U^*=UAU^*.$$

Let $S=U^*$. Then $\|S\|\leqq 1$, $B=S^*AS$ and $T=(TT^*)^{1/2}U^*=A^{1/2}S$. Hence the proof is completed.

As a special case of the theorem, we have the following

Corollary ([1]). *Let T be a contraction and A the strong limit of the sequence $\{T^{*n}T^n\}$. Then $A^{1/2}T$ is a hyponormal operator.*

Proof. The assertion is clear, because $A=T^*AT$ by the definition of A.

The following lemma is a generalization of a result in [1].

Lemma. *In the theorem, suppose that S is completely non-unitary.* Then $T=A^{1/2}S$ is normal if and only if $A=0$.

Proof. 'If part' is trivial. Assume that T is normal. Then we see from (1) that

(2) $$B-A^{1/2}SS^*A^{1/2}=A-A^{1/2}SS^*A^{1/2}=0.$$

Thus, $A=B$ and $S^*AS=A$. By (2), we have

$$A^{1/2}(I-SS^*)A^{1/2}=0.$$

Since $I-SS^*\geqq 0$, we see that $(I-SS^*)A=0$. Thus we have

$$A=SS^*A, \qquad AS=SS^*AS=AS$$

and so

$$(I-S^*S)A=A-S^*SA=A-S^*AS=0.$$

The closed subspace $[AH]^{\perp\perp}$ reduces S and S is unitary on $[AH]^{\perp\perp}$. Therefore $A=0$, for S is completely non-unitary.

Reference

[1] T. Yoshino: Hyponormal operators in von Neumann algebras (to appear).