111. *On the σ-Socle of a Module*

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(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1975)

Let R be a ring with identity and let σ be a left exact radical on R-mod such that $T(\sigma)$ is a TTF class. The purpose of this paper is to show that, for any module M, the sum of all σ-simple submodules of M coincides with the intersection of all σ-essential submodules of M. In case $\sigma=1$, i.e., $T(\sigma)=R$-mod, the above result means the so-called Sandomierski-Kasch's characterization of the socle of a module (see [1, p. 62]).

Let σ be a left exact preradical on the category R-mod of unital left R-modules. Then the class $T(\sigma)=\{M \mid \sigma(M)=M\}$ is closed under submodules, quotients and direct sums. The modules in $T(\sigma)$ are called *σ-torsion*. A submodule L of a module M with $M/L \in T(\sigma)$ is called *σ-open* in M. If L is both σ-open and essential in M, we say that L is *σ-essential* in M. The *σ-socle* of a module $M\neq 0$, denoted by σ-soc (M), is defined as the intersection of all σ-essential submodules of M. If $M=0$ we define $M=\sigma$-soc (M). A module S is called *σ-simple* if for any σ-open submodule A of S, either $A=S$ or $A=0$.

Lemma. *If S is a σ-simple submodule of M, then $S\subseteq\sigma$-soc (M).*

Proof. We may assume $S\neq 0$. If L is a σ-essential submodule of M, $S\cap L\neq 0$ and $S\cap L$ is σ-open in S, since $S/(S\cap L)\cong(S+L)/L\subseteq M/L \in T(\sigma)$. Thus $S\cap L=S$ and so $S\subseteq L$.

A module M is *σ-semisimple* if every σ-open submodule of M is a direct summand of M. From [2], we quote the following facts:

(A) A σ-torsion module is σ-semisimple if and only if it is semisimple.

(B) If M is σ-semisimple, and N is any submodule of M, then M/N is σ-semisimple.

Now we assume moreover that σ is a left exact radical such that $T(\sigma)$ is a TTF class, i.e., $T(\sigma)$ is closed additionally under extensions and direct products. In this case, the corresponding topology $\mathcal{F}=\{I \mid I$ is a left ideal with $R/I \in T(\sigma)\}$ has a smallest member U. U is idempotent and $T(\sigma)=\{M \mid UM=0\}$.

Theorem. *If σ is a left exact radical such that $T(\sigma)$ is a* TTF *class, then for any module M, σ-soc $(M)=\Sigma\{S\subseteq M \mid S$ is σ-simple$\}$. Moreover σ-soc (M) is a direct sum of σ-simple submodules.*

Proof. We show only the last assertion holds, then the former follows from Lemma. Put $N=\sigma\text{-soc}(M)$, and take any σ-open submodule K in N. Let K' be a complement of K in M, then $K+K'=K\oplus K'$ is essential in M. By the definition of N, M/N can be embedded in a direct product of σ-torsion modules. Thus $M/N\in \boldsymbol{T}(\sigma)$. Since we have the exact sequence $0\to N/K\to M/K\to M/N\to 0$, $M/K\in \boldsymbol{T}(\sigma)$ and so $M/(K+K')\in \boldsymbol{T}(\sigma)$. Therefore $K+K'$ is σ-essential in M, and so we obtain $N\subseteq K+K'$. By modularity

$$N=N\cap(K+K')=K\oplus(N\cap K').$$

This shows that N is σ-semisimple. It is immediate that $UM\subseteq N$. Since $N/UM\subseteq M/UM\in \boldsymbol{T}(\sigma)$, UM is σ-open in N. Thus $N=UM\oplus X$, where X is semisimple by using (A) and (B). It remains to show that UM is σ-simple. For any σ-open submodule C of UM, $UM=U(UM)\subseteq C$. Thus we have $UM=C$.

Remark. Note that the theorem is false if the assumption that $\boldsymbol{T}(\sigma)$ is a TTF class is dropped. Let K be a field and $R=\prod_{\alpha\in\Lambda}K_\alpha$ where $K_\alpha=K$ for all $\alpha\in\Lambda$ and Λ is a fixed infinite indexed set. Define a left exact radical σ on R-mod by the corresponding topology $\mathcal{F}=\{\prod_{\beta\in\Gamma}K_\beta \mid \Gamma$ is a subset of Λ with finite complement$\}$. Rubin [2] showed that $\sigma\text{-soc}(R)=R$. Now we show that any σ-simple ideal A $(\neq 0)$ of R is of the form K_α. For some α, $K_\alpha\subseteq A$ and so we may write $A=K_\alpha\oplus B$, where $B\subseteq\prod_{\beta\neq\alpha}K_\beta$. Since $A/B\cong K_\alpha\in \boldsymbol{T}(\sigma)$, B is σ-open in A. Thus $B=0$ as desired. Therefore the sum of all σ-simple ideals in R is $\bigoplus_{\alpha\in\Lambda}K_\alpha\neq R$.

References

[1] J. Lambek: Lectures on Rings and Modules. Blaisdell (1966).
[2] R. A. Rubin: Semi-simplicity relative to kernel functors. Canad. J. Math., **26**, 1405–1411 (1974).