78. On the Summability of Taylor Series of the Regular Function of Bounded Type in the Unit Circle

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1. Introduction. The object of this note is to introduce a new summation process, by means of which Taylor series of the regular function of bounded type in $|z|<1$ can be summable on $|z|=1$. The details of proofs will be published elsewhere in near future.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be the regular function of bounded type in the unit circle. In general, the series $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is not Cesàro-summable. In fact, put

$$f(z)=\exp\left(\frac{\alpha}{2}\cdot\frac{1+z}{1-z}\right)=\sum_{n=0}^{\infty} a_n z^n \qquad \text{for } \alpha>0,\ |z|<1,$$

which is the regular function of bounded type in the unit circle. Then we have

$$a_n=\exp\left(2\sqrt{\alpha n}+O(\ln n)\right)$$

([1] pp. 107–108). Since there exists no $k>-1$ such that $a_n=o(n^k)$, the series $\sum_{n=0}^{\infty} a_n z^n$ is not Cesàro-summable on $|z|=1$ (2 p. 78).

2. Notations and definitions. As usual, for $k>-1$, $|x|<1$, we put

$$\frac{1}{(1-x)^{k+1}}=\sum_{n=0}^{\infty} A_n^{(k)}\cdot x^n, \qquad \frac{1}{(1-x)^{k+1}}\cdot\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} S_n^{(k)}\cdot x^n,$$

where $S_n^{(k)}=\sum_{i=0}^{n} a_i A_{n-i}^{(k)}$, $C_n^{(k)}=S_n^{(k)}/A_n^{(k)}$. If $C_n^{(k)}\to s$ as $n\to\infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ is Cesàro-summable (C, k) to s. For brevity, we write

$$\sum_{n=0}^{\infty} a_n = s(C, k).$$

Generalizing this Cesàro-summation, we introduce following summation process. For $k>-1$, $\alpha>0$ and $|x|<1$, we put

$$\frac{1}{(1-x)^{k+1}}\cdot\exp\left(\frac{\alpha}{1-x}\right)=\sum_{n=0}^{\infty} b_n(k,\alpha)\cdot x^n,$$

$$\frac{1}{(1-x)^{k+1}}\cdot\exp\left(\frac{\alpha}{1-x}\right)\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} S_n(k,\alpha)x^n,$$

where $S_n(k,\alpha)=\sum_{i=0}^{n} a_i b_{n-i}(k,\alpha)$, $C_n(k,\alpha)=S_n(k,\alpha)/b_n(k,\alpha)$. *If $C_n(k,\alpha)\to s$ as $n\to\infty$, we say that the series $\sum_{n=0}^{\infty} a_n$ is summable (C, k, α) to s.* For brevity, we write

$$\sum_{n=0}^{\infty} a_n = s(C, k, \alpha).$$

We call this summation (C, k, α)-summation.

3. Abelian theorems. The interrelation between Cesàro-summation and (C, k, α)-summation is shown in

Theorem 1. *Let* $f(x)=\sum_{n=0}^{\infty} a_n x^n$ *be convergent for* $|x|<1$, *where* $\{a_n\}$ *are real numbers. Then following inequalities hold;*

$$\varliminf_{n\to\infty} C_n^{(k)} \leqq \varliminf_{n\to\infty} C_n(k,\alpha) \leqq \varliminf_{n\to\infty} C_n(k',\alpha) \leqq \varliminf_{x\to 1} f(x) \leqq$$
$$\varlimsup_{x\to 1} f(x) \leqq \varlimsup_{n\to\infty} C_n(k',\alpha) \leqq \varlimsup_{n\to\infty} C_n(k,\alpha) \leqq \varlimsup_{n\to\infty} C_n^{(k)},$$

where $k'>k>-1$, $\alpha>0$.

Taking the real and imaginary parts respectively, we get

Corollary 1. [A] *If* $\sum_{n=0}^{\infty} a_n = s(C, k)$, *then* $\sum_{n=0}^{\infty} a_n = s(C, k, \alpha)$, *where* $k>-1$, $\alpha>0$. [B] *If* $\sum_{n=0}^{\infty} a_n = s(C, k, \alpha)$, *then* $\sum_{n=0}^{\infty} a_n = s(C, k', \alpha)$ *where* $k'>k>-1$, $\alpha>0$.

Corollary 2. *If* $\sum_{n=0}^{\infty} a_n = s(C, k, \alpha)$ $(k>-1, \alpha>0)$, *then*

$$\lim_{x\to 1-0} \sum_{n=0}^{\infty} a_n x^n = s.$$

We can extend Corollary 2 as follows;

Theorem 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be convergent for* $|z|<1$, *where* $\{a_n\}$ *are complex numbers. If* $\sum_{n=0}^{\infty} a_n = s(C, k, \alpha)$ $(k>-1, \alpha>0)$, *then* $f(z)$ *tends uniformly to* s *as* $z\to 1$, $z\in D(K,d)$, *where* $D(K,d)$ *is the "cuspidal domain" with its cusp at* $z=1$:

$$\{|z-1|\leqq d\} \cup \{z=re^{i\theta} : |\theta| \leqq K(1-r)^{3/2}\},$$

d: *a sufficiently small positive constant and* K: *a positive finite constant.*

4. Tauberian theorems. We denote by N the class of regular functions $f(z)$ of bounded type in $|z|<1$. Then we have

$$A(f)=\lim_{r\to 1} 1/2\pi \int_0^{2\pi} \ln^+ |f(\mathrm{r}e^{i\theta})|\, d\theta < +\infty.$$

Next Tauberian theorem holds;

Theorem 3. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n \in N$ *and let* $\alpha=2A(f)>0$. *Suppose that* $f(z)$ *has the angular limit*: $f(e^{i\theta_0})$ *at* $z=e^{i\theta_0}$ *such that*

$$f(z)=f(e^{i\theta_0})+o(\sqrt{|z-e^{i\theta_0}|}) \qquad \text{as } z\to e^{i\theta_0},\ z\in S,$$

where S *is Stolz domain with its vertex at* $z=e^{i\theta_0}$. *If*

$$(*) \qquad \varlimsup_{r\to 1} (1-r)\ln^+ M(r,\eta,\theta_0) < \alpha$$

for sufficiently small $\eta>0$, *where*

$$M(r,\eta,\theta_0)=\max_{|h|\leqq\eta} \left| 1/h\cdot\int_{\theta_0}^{\theta_0+h} |f(re^{i\theta})|\, d\theta \right|,$$

then $\sum_{n=0}^{\infty} a_n e^{in\theta_0}$ *is summable* (C, k, α) *to* $f(e^{i\theta_0})$ *for* $k>1/2$.

As an immediate consequence, we have

Corollary 3. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n \in N$ *and let* $\alpha=2A(f)>0$. *If* $f(z)$ *is regular at* $z=e^{i\theta_0}$, *then* $\sum_{n=0}^{\infty} a_n e^{in\theta_0}$ *is summable* (C, k, α) *to* $f(e^{i\theta_0})$ *for* $k>1/2$.

In Theorem 3, the condition (*) can be replaced by more practical one;

Corollary 4. *In Theorem* 3 *the condition* (*) *can be replaced by*

$$\int_{\theta_0-\eta}^{\theta_0+\eta} \phi(\ln^+ |f(re^{i\theta})|)d\theta = 0(1) \qquad \text{as } r\to 1,$$

for sufficiently small $\eta>0$, *where* $\phi(x)$ *is a non-negative function of* x $(0\leqq x<\infty)$ *such that* $\phi(x)/x$ *is non-decreasing and tends to* ∞ *as* $x\to\infty$.

References

[1] I. I. Priwalow: Randeigenschaften analytischer Funktionen (1956). Berlin.
[2] A. Zygmund: Trigonometric Series (Second Edition) Vol. I (1968). Cambridge University Press.