112. On Nonlinear Evolution Equations with a Difference Term of Subdifferentials

By Yoshiharu KOI*) and Jiro WATANABE**)

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1976)

1. Introduction. Let φ be a lower semicontinuous convex function on a real Hilbert space H into $(-\infty, \infty]$ with $\varphi \not\equiv \infty$. We define the subdifferential $\partial\varphi$ of φ by $\partial\varphi(v) = \{w \in H; \varphi(u) \geqq \varphi(v) + (u-v, w)$ for all $u \in H\}$ for each $v \in H$. We set $D(\varphi) = \{v \in H; \varphi(v) < \infty\}$ and $D(\partial\varphi) = \{v \in H; \partial\varphi(v) \neq \phi\}$. For each $v \in D(\partial\varphi)$, $\partial\varphi^0(v)$ denotes the uniquely determined element of least norm in $\partial\varphi(v)$. Let ψ be another lower semicontinuous convex function on H into $(-\infty, \infty]$ with $\psi \not\equiv \infty$. We write $J(v) = \varphi(v) - \psi(v)$ for all $v \in D(\varphi)$.

Let us consider the nonlinear evolution equation

(1) $$du(t)/dt + \partial\varphi(u(t)) - \partial\psi(u(t)) \ni f(t), \qquad t > 0$$

with the initial condition

(2) $$u(0) = a.$$

We assume that φ and ψ satisfy the following conditions (I) and (II).

(I) $D(\varphi) \subsetneqq D(\partial\psi)$.

(II) *If B is a bounded subset of $D(\varphi)$ such that J is bounded from above on B, then B is relatively compact in H and $\partial\psi^0$ is bounded on B.*

In this paper we will show that conditions (I) and (II) are sufficient for the local existence of solutions of the initial value problem (1), (2). We will also prove extension theorems for solutions. Recently M. Otani dealt with the initial value problem (1), (2) under certain assumptions which are sufficient for the global existence of solutions. Otani's conditions on φ and ψ are different from ours.

In another paper we will discuss applications to the initial boundary value problems for nonlinear parabolic equations treated by Tsutsumi [2].

2. Local existence theorem. Let I be an interval in $(-\infty, \infty)$. We denote by $L^2_{\text{loc}}(I; H)$ the space of all H-valued strongly measurable functions g on I such that

$$\int_K \|g(t)\|^2 \, dt < \infty$$

for all compact interval K in I. When $-\infty < r < s < \infty$, we write $L^2(r, s; H)$ instead of $L^2_{\text{loc}}([r, s]; H)$.

Definition. Let u be an H-valued continuous function on $[0, S)$

*) Saitama Prefectural Toyooka Senior High School.
**) The University of Electro-Communications.

and let $f \in L^2(0, T; H)$, where $0 < S \leqq T$. We say u is a *strong solution* of the problem (1), (2) on $[0, S)$, if u has the following properties:

(i) $u(t) \in D(\partial\varphi)$ for almost all $t \in (0, S)$;

(ii) there exist g and h in $L^2_{\mathrm{loc}}([0, S); H)$ such that $g(t) \in \partial\varphi(u(t))$ and $h(t) \in \partial\psi(u(t))$ almost everywhere in $(0, S)$ and

$$(3) \qquad u(t) = a + \int_0^t [f(s) - g(s) + h(s)]ds$$

holds for all $t \in [0, S)$.

Remark 1. If u is a strong solution of the problem (1), (2) on $[0, S)$, then $\varphi(u(t))$ and $\psi(u(t))$ are absolutely continuous on $[0, r]$ for any $r \in (0, S)$ and

$$(4) \qquad J(u(t)) - J(a) = \int_0^t (f(s) - u'(s), u'(s))ds$$

holds for all $t \in [0, S)$, where $u'(s)$ denotes $du(s)/ds$. Cf. Brezis [1, Lemme 3.3].

Now we state a local existence theorem.

Theorem 1. *Suppose conditions* (I) *and* (II) *are satisfied. Let* $a \in D(\varphi)$ *and* $f \in L^2(0, T; H)$. *Then there exists at least one strong solution* u *of the problem* (1), (2) *on* $[0, S)$ *for some* $S \in (0, T]$.

Remark 2. The strong solution u of the problem (1), (2) is not necessarily unique. For an example of non-uniqueness, let us take $H = \boldsymbol{R}$ (set of real numbers), $\varphi \equiv 0$ and $\psi(v) = |v|^q/q$ ($v \in \boldsymbol{R}$), where $1 < q < 2$. Then φ and ψ satisfy conditions (I) and (II). When $f \equiv 0$, the equation (1) reduces to the ordinary differential equation $u' = |u|^{q-2}u$. It is well-known that the solution u of this equation with the initial condition $u(0) = 0$ is not unique.

From now on in this paper we always assume that conditions (I) and (II) are satisfied and we fix arbitrarily an $a \in D(\varphi)$ and an $f \in L^2(0, T; H)$.

3. Proof of Theorem 1. For all $\lambda > 0$ and all $v \in H$ we write $J_\lambda(v) = \varphi(v) - \psi_\lambda(v)$ where $\psi_\lambda(v) = \min_{y \in H} [\psi(y) + \|y - v\|^2/(2\lambda)]$. Note that $J(v) \leqq J_\lambda(v) \leqq J_\mu(v)$ when $v \in D(\varphi)$ and $0 < \lambda \leqq \mu$.

Let us consider the initial value problem

$$(5) \qquad du_\lambda/dt + \partial\varphi(u_\lambda) - \partial\psi_\lambda(u_\lambda) \ni f(t), \qquad 0 < t < T$$

$$(6) \qquad u_\lambda(0) = a.$$

For each $\lambda > 0$ there exists a uniquely determined strong solution u_λ of the problem (5), (6). See, e.g., Brezis [1, Proposition 3. 12] or Watanabe [3, Theorem 1.1]. Then, by (4), we get

$$(7) \qquad \int_0^t \|u'_\lambda\|^2 ds + J_\lambda(u_\lambda(t)) = \int_0^t (f, u'_\lambda)ds + J_\lambda(a), \qquad 0 \leqq t \leqq T.$$

Lemma 1. *Let* B_r *be the set* $\{u_\lambda(t) \in H; 0 < \lambda < 1,\ 0 \leqq t \leqq T$ *and* $J_\lambda(u_\lambda(t)) \geqq r\}$ *where* r *is a real number. Then* B_r *is relatively compact in* H *and* $\partial\psi^0$ *is bounded on* B_r.

Proof. From (7) we can show easily that B_r is bounded and that J is bounded from above on B_r. Then the assertion follows from the assumption (II).

Lemma 2. *If $r<J(a)$, then there exists an $S\in(0,T]$ such that*

(8) $$J_\lambda(u_\lambda(t))>r \quad \text{when } 0\leqq t\leqq S \text{ and } 0<\lambda<1.$$

Proof. Assume that there exists a sequence $\{(t_n,\lambda_n)\}_n$ in $(0,T]\times(0,1)$ satisfying $\sup_n J_{\lambda_n}(u_{\lambda_n}(t_n))\leqq r$ and $\lim_{n\to\infty} t_n=0$. Then $J_{\lambda_n}(u_{\lambda_n}(0))\geqq J(a)>r\geqq J_{\lambda_n}(u_{\lambda_n}(t_n))$. Since $J_{\lambda_n}(u_{\lambda_n}(t))$ is continuous in $[0,T]$ (see Remark 1), we may assume that $J_{\lambda_n}(u_{\lambda_n}(t_n))=r$ is satisfied for all n. Then, by (7), we get easily

$$C\equiv\sup_n\int_0^{t_n}\|u'_{\lambda_n}(s)\|^2\,ds<\infty$$

and so $\|u_{\lambda_n}(t_n)-a\|\leqq\sqrt{t_nC}$, which implies $\lim_{n\to\infty}u_{\lambda_n}(t_n)=a$ because $\lim_{n\to\infty}t_n=0$ by assumption. Since $\sup_n J(u_{\lambda_n}(t_n))\leqq r$, it follows from (II) that $\{\partial\psi^0(u_{\lambda_n}(t_n))\}_n$ is bounded. Hence we have $\sup_n\|\partial\psi_{\lambda_n}(u_{\lambda_n}(t_n))\|<\infty$, because $\|\partial\psi_{\lambda_n}(v)\|\leqq\|\partial\psi^0(v)\|$ for all $v\in D(\partial\psi)$. Therefore we obtain

(9) $$\lim_{n\to\infty}(\partial\psi_{\lambda_n}(u_{\lambda_n}(t_n)),\,a-u_{\lambda_n}(t_n))=0.$$

On the other hand we have

$$\begin{aligned}J(a)-r&\leqq J_{\lambda_n}(a)-J_{\lambda_n}(u_{\lambda_n}(t_n))\\&\leqq\varphi(a)-\varphi(u_{\lambda_n}(t_n))-(\partial\psi_{\lambda_n}(u_{\lambda_n}(t_n)),\,a-u_{\lambda_n}(t_n)).\end{aligned}$$

Combining this with (9), we get

$$J(a)-r\leqq\varphi(a)-\liminf_{n\to\infty}\varphi(u_{\lambda_n}(t_n)).$$

While the left-hand side of this inequality is positive by assumption, the right-hand side is non-positive by the lower semicontinuity of φ. This is a contradiction. The proof is complete.

End of the proof of Theorem 1. Let $r<J(a)$. Then, by Lemma 2, we can choose an $S\in(0,T]$ satisfying (8). We set $B=\{u_\lambda(t)\in H;\ 0<\lambda<1 \text{ and } 0\leqq t\leqq S\}$. Then it follows from Lemma 1 that B is relatively compact in H and that $\|\partial\psi_\lambda(u_\lambda(t))\|\leqq\sup_{v\in B}\|\partial\psi^0(v)\|<\infty$ holds for all $\lambda\in(0,1)$ and all $t\in[0,S]$. We write

(10) $$g_\lambda(t)=f(t)+\partial\psi_\lambda(u_\lambda(t))-u'_\lambda(t),\qquad \lambda>0,\ 0\leqq t\leqq S.$$

Since $\{u'_\lambda\}_{0<\lambda<1}$ is bounded in $L^2(0,S;H)$, $\{g_\lambda\}_{0<\lambda<1}$ is also bounded in $L^2(0,S;H)$. Then we can choose a sequence $\lambda_n\searrow 0$ with the following properties: (i) u_{λ_n} converges to a continuous function u uniformly on $[0,S]$ and (ii) $\partial\psi_{\lambda_n}(u_{\lambda_n})$ (respectively g_{λ_n}) converges weakly to an h (respectively a g) in $L^2(0,S;H)$. Then we have $h(t)\in\partial\psi(u(t))$ and $g(t)\in\partial\varphi(u(t))$ almost everywhere in $(0,S)$ (see Brezis [1, Proposition 2.16]). Integrating with respect to t both sides of (10) with $\lambda=\lambda_n$ and making $n\to\infty$, we obtain (3) for all $t\in[0,S]$. Hence u is a strong solution of (1), (2) on $[0,S]$. The proof is complete.

4. Extension Theorem. Throughout this section we assume that

u is a strong solution of the initial value problem (1), (2) on $[0, S)$ for some $S \in (0, T)$. We have the following extension theorem.

Theorem 2. *If $J(u(t))$ is bounded from below on $[0, S)$, then u can be continued to the right of S as a strong solution of the problem* (1), (2).

Proof. By (4) we get easily $u' \in L^2(0, S; H)$. Hence $u(t)$ converges to a v in H as $t \nearrow S$. In view of Theorem 1 it is sufficient for the proof of the theorem to show $v \in D(\varphi)$. By (4) again, we can show easily that $J(u(t))$ is bounded from above in $[0, S)$. Hence, by the assumption (II), $\partial\psi^0(u(t))$ is bounded in $[0, S)$. Then, since

$$\begin{aligned}\varphi(u(t)) &= J(u(t)) + \psi(u(t)) \\ &\leqq J(u(t)) + \psi(a) + (u(t)) - a, \partial\psi^0(u(t))\end{aligned}$$

for all $t \in [0, S)$, it follows that $\varphi(u(t))$ is bounded from above in $[0, S)$. Therefore $\varphi(v) \leqq \liminf_{t \nearrow S} \varphi(u(t)) < \infty$, that is, $v \in D(\varphi)$. This completes the proof.

In view of Theorem 2 and Remark 1, we can prove easily the following

Theorem 3. *We have $\lim_{t \nearrow S} J(u(t)) = -\infty$, if and only if u cannot be continued to the right of S as a strong solution of the problem* (1), (2).

References

[1] H. Brezis: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland, Amsterdam (1973).

[2] M. Tsutsumi: Existence and nonexistence of global solutions for nonlinear parabolic equations. Publ. RIMS, Kyoto Univ., **8**, 211–229 (1972/73).

[3] J. Watanabe: On certain nonlinear evolution equations. J. Math. Soc. Japan, **25**, 446–463 (1973).