9. *Remarks on Ideals of Bounded Krull Prime Rings*

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1. Introduction. Throughout this paper all notations and all terminologies are the same as in [6] and [7]. Let R be a bounded Krull prime ring with the non-empty set of minimal non-zero prime ideals, $M(p)$ say, and let Q be the quotient ring of R. Then $R=\bigcap R_P$ ($P \in M(p)$) and each R_P is a noetherian, local, Asano order in Q. Let F be any right additive topology. We denote by R_F the ring of quotients with respect to F (cf. § 7 of [8]). Let F and F' be right additive topologies of integral right R-ideals. If $R_F=R_{F'}$, then they are said to be *equivalent*.

The aim of this paper is to prove the following theorems.

Theorem A. *Let $P_1, \cdots, P_k \in M(p)$ and let $\bar{I}_i$ be any right R_{P_i}-ideals ($1 \leqq i \leqq k$). Then there exists a unit x in Q such that $xR_{P_i}=\bar{I}_i$ ($1 \leqq i \leqq k$) and $x \in R_{P_j}$ for all $P_j \in M(p)$ with $P_j \neq P_i$.*

Theorem B. *Let I be any right R-ideal and let a be any regular element in I. Then there exists an element b in I such that $I^*=(aR+bR)^*$.*

Theorem C. *Let F be any right additive topology of integral right R-ideals. Then*

(1) *If $F \cap M(p)=\phi$, then $F^*=\{I \mid I^*=R\}$ is a unique maximal element in the set of right additive topologies equivalent to F, and $R_F=R$.*

(2) *If $F \cap M(p) \neq \phi$, then $F^*=\{I \mid I^* \supseteqq P_1^{n_1} \cdots P_k^{n_k}$, where $P_i \in F \cap M(p)\}$ is a unique maximal element in the set of right additive topologies equivalent to F. If $F(p)=M(p)$, where $F(p)=F \cap M(p)$, then $R_F=Q$, and if $M(p) \supsetneqq F(p)$, then $R_F=\bigcap R_P$ ($P \in M(p)-F(p)$).*

2. The proofs of Theorems. (a) First we shall prove Theorem A. To this we let $F(p)=\{P_i \mid 1 \leqq i \leqq k\}$ and let $I=\bar{I}_1 \cap \cdots \cap \bar{I}_k \cap \bigcap_j R_{P_j}$ ($P_j \in M(p)-F(p)$). Then it is clear that I is a right R-ideal. By Lemma 2.1 of [5] $IR_{P_i}=\bar{I}_i$ and $IR_{P_j}=R_{P_j}$. Let $A=P_1 \cap \cdots \cap P_k$. Then there exists a regular element c in Q such that $IR_A=cR_A$ by Lemma 3.3 of [6] and so $IR_{P_i}=cR_{P_i}$ ($1 \leqq i \leqq k$). If $c \in R_{P_j}$ for all $P_j \in M(p)-F(p)$, then c is an element satisfying the assertion. If $c \notin R_{P_j}$ for some $P_j \in M(p)-F(p)$, then there are only finitely many elements $P_{k+1}, \cdots, P_{k+l}$ in $M(p)$ such that $c \notin R_{P_{k+j}}$ ($1 \leqq j \leqq l$). Let $B=P_{k+1} \cap \cdots \cap P_{k+l}$. Then it follows that $Q=\varinjlim (P_{k+1}R_B)^{-n_1} \cdots (P_{k+l}R_B)^{-n_l}$ by Proposition 1.2, Lemma 3.3 of [6]. So there are non-negative integers $n_1, \cdots, n_l$ such that $cC \subsetneqq R_B$, where $C = P_{k+1}^{n_1} \cdots P_{k+l}^{n_l}$. Let d be any element in $C \cap C(A)$, which is non-empty by Lemma 3.1 of [6]. If $x = cd$, then it follows that $xR_{P_i} = \bar{I}_i$ $(1 \leqq i \leqq k)$ and $x \in R_{P_j}$ for each $P_j \in M(p) - F(p)$.

(b) Next we shall prove Theorem B. It suffices to prove the result in case $I \subsetneqq R$. There are finitely many $P_1, \cdots, P_k$ in $M(p)$ such that $aR_{P_i} \subsetneqq R_{P_i}$ $(1 \leqq i \leqq k)$. Let $A = P_1 \cap \cdots \cap P_k$. Then, by Corollary 3.5 of [1], there exists an element b in $IR_A = aR_A + bR_A = (aR + bR)R_A$. Since R_A is the quotient ring of R with respect to $C(A)$, we may assume that $b \in I$. It follows that $IR_{P_i} = (aR + bR)R_{P_i}$ $(1 \leqq i \leqq k)$ and that $R_{P_j} \supseteqq IR_{P_j} \supseteqq (aR + bR)R_{P_j} = R_{P_j}$ for each $P_j \in M(p)$ with $P_j \neq P_i$. So $\cap IR_P = \bigcap (aR + bR)R_P$ $(P \in M(p))$. Hence $I^* = (aR + bR)^*$ by Proposition 1.10 of [6].

(c) Finally we shall prove Theorem C.

($*$) First we shall prove that if there exists $J \in F$ such that $J^* \neq R$, then $F \cap M(p)$ is non-empty. To prove this let I $(\neq R)$ be any element in F such that it is a maximal element in $F'_r(R)$. Then it follows that $I \supseteqq P^n$ for some $P \in M(p)$ by Lemma 6 of [7]. If $P^{n-1} \not\subseteq I$ and $n > 1$, then we have $P = (I \cup {}^*(P^{n-1})^* \circ P = I \circ P \cup {}^*(P^n)^* \subsetneqq I$, a contradiction. Hence $P \subseteqq I$. We shall prove that $P \in F$. By Lemma 9 of [7], $I \cap C(P) = \phi$. Let A be any maximal element in the family $\{L \mid L \supseteqq I,\ L \cap C(P) = \phi,\ L\colon$ integral right R-ideal$\}$. Then $\bar{A} = A/P$ is a maximal complemented right ideal of $\bar{R} = R/P$ in the sense of Goldie. Further let B $(\supseteqq P)$ be any right ideal of R such that $\bar{B}$ is maximal complemented in $\bar{R}$. Then, by Theorem 8 of [2] and Theorem 3.7 of [3], there exist uniform elements $\bar{u}, \bar{v}$ in $\bar{R}$ such that $(\bar{0} : \bar{u})_r = \bar{A}$ and $(\bar{0} : \bar{v})_r = \bar{B}$, that is, $A = u^{-1}P$ and $B = v^{-1}P$. By Lemma 3.1 of [4], $\bar{u}\bar{R}$ and $\bar{v}\bar{R}$ are subisomorphic. Further $R/A = R/u^{-1}P \cong (uR + P)/P = \bar{u}\bar{R}$ and $R/B = \bar{v}\bar{R}$. Since $A \in F$, R/A is F-torsion and so R/B is also F-torsion. Thus we have $B \in F$. By Theorem 2.3 of [3], $\bar{0} = \bar{A}_1 \cap \cdots \cap \bar{A}_n$, where $\bar{A}_i$ are maximal complemented in $\bar{R}$, that is, $P = A_1 \cap \cdots \cap A_n$. Since $A_i \in F$, we get $P \in F$, as desired.

(1) We let $F^* = \{I \mid I^* = R\}$. Then it is a right additive topology. If $F \cap M(p) = \phi$, then $F \subseteqq F^*$ by ($*$) and $R_F = R = R_{F^*}$. It is evident that F^* is a unique maximal element in the set of right additive topologies equivalent to F.

(2) Let $F(p) = F \cap M(p)$. First we shall prove that $F^* = \{I \mid I^* \supseteqq P_1^{n_1} \cdots P_k^{n_k},\ P_i \in F(p)\}$ is a right additive topology. If $I \in F^*$ and $r \in R$, then we have $(r^{-1}I)^* = r^{-1}I^*$ by Lemma 2.3, Theorem 2.6 and Proposition 1.10 of [6]. So it follows that $r^{-1}I \in F^*$. If J is a right ideal such that $a^{-1}J \in F^*$ for some $I \in F^*$ and any $a \in I$, then we must prove that $J \in F^*$. If $J^* \supseteqq I$, then $J \in F^*$ and so we assume that $J^* \not\supseteqq I$. It suffices to prove the result in case J^* is irreducible in $F'_r(R)$. Let a be any element in

I but not in J^*. Since $a^{-1}J \in F^*$, there exist $P_1, \cdots, P_k$ in $F(p)$ such that $a^{-1}J^* = (a^{-1}J)^* \supsetneqq P_1^{n_1} \cdots P_k^{n_k}$, that is $(aR)P_1^{n_1} \cdots P_k^{n_k} \subsetneqq J^*$. Hence $(P_1^{n_1} \cdots P_k^{n_k})^n \subsetneqq J^*$ by Lemma 6 of [7] and so $J \in F^*$. This proves that F^* is a right additive topology. Let $P_1, \cdots, P_k \in F(p)$. Then it is well known that $P_1^{n_1} \cdots P_k^{n_k} \in F$ so that $R_F \supsetneqq R_{F^*}$. To prove that $F \subsetneqq F^*$, we let I be any element in F. If $I^* = R$, then $I \in F^*$. If $I^* \neq R$ and I^* is irreducible in $F'_r(R)$, then there exist $P \in M(p)$ and n such that $P^n \subsetneqq I^*$. Let I_0 be any maximal elements in $F'_r(R)$ such that $I_0 \supsetneqq I^*$ and $I_0 \neq R$. Since $I_0 \supsetneqq P^n$, it follows that $P \in F(p)$ from the proof of (*). Therefore $I \in F^*$. If I^* is reducible, then $I^* = I_1 \cap \cdots \cap I_k$, where $I_i \in F \cap F'_r(R)$ and I_i are irreducible. Hence $I_i \supsetneqq P_i^{n_i}$ for some $P_i \in F(p)$ so that $I^* \supsetneqq P_1^{n_1} \cdots P_k^{n_k}$. Therefore $I \in F^*$ and so $F \subsetneqq F^*$. This implies that $R_F \subsetneqq R_{F^*}$, and hence we have $R_F = R_{F^*} = \varinjlim (P_1^{n_1} \cdots P_k^{n_k})^{-1}$, where $P_i \in F(p)$. By using Theorem B it follows that F^* is a unique maximal element in the set of right additive topologies equivalent to F. Finally if $F(p) = M(p)$, then we get $R_F = Q$ by Lemma 1.6 of [6]. Suppose that $M(p) \supsetneqq F(p) \neq \phi$. Let P be any element in $M(p) - F(p)$ and let $P_1, \cdots, P_k$ be any elements in $F(p)$. Then $(P_1^{n_1} \cdots P_k^{n_k})^{-1} \subsetneqq R_P$ by Lemma 11 of [7]. Hence R_F is contained in the ring $T = \bigcap R_P$ $(P \in M(p) - F(p))$. Conversely let x be any element in T. Then there exists an ideal $B_P (\not\subseteq P)$ such that $xB_P \subsetneqq R$ for any $P \in M(p) - F(p)$. Write $B = \sum B_P$. Then we have $(xR + R)B^* \subsetneqq (xB + B)^* \subsetneqq R$ so that $x \in B^{-1}$. If $B^* = R$, then $x \in R$. If $B^* \neq R$ and if $B^* = (P_1^{n_1})^* \circ \cdots \circ (P_l^{n_l})^*$, where $P_i \in M(p)$, then $P_i \in F(p)$. Hence $x \in B^{-1} = (P_1^{n_1} \cdots P_l^{n_l})^{-1} \subsetneqq R_F$ and thus we have $R_F = \bigcap R_P$ $(P \in M(p) - F(p))$.

Corollary 1. *Let R be a bounded Krull prime ring and let F be any right additive topology consists of integral right R-ideals. Then R_F is a bounded Krull prime ring, or a simple and artinian ring.*

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