10. *Asymptotic Properties of Functional Differential Equations with a General Deviating Argument*

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1. Introduction. We consider the n-th order functional differential equation

(A) $$L_n x(t)+f(t, x(g(t)))=0,$$

where the differential operator L_n is recursively defined by

$$L_1 x(t)=x'(t), \quad L_i x(t)=(p_{i-1}(t)L_{i-1}x(t))', \qquad 2\leqq i\leqq n.$$

The following assumptions are made without further mention:

(a) Each $p_i(t)$ is continuous and positive on $[a, \infty)$, and

$$\int_a^\infty \frac{dt}{p_i(t)}<\infty, \qquad 1\leqq i\leqq n-1;$$

(b) $g(t)$ is continuous on $[a, \infty)$ and $\lim_{t\to\infty} g(t)=\infty$;

(c) $f(t, x)$ is continuous on $[a, \infty)\times(-\infty, \infty)$ and $|f(t, x)|\leqq\omega(t, |x|)$ for $(t, x)\in[a, \infty)\times(-\infty, \infty)$, where $\omega(t, r)$ is continuous on $[a, \infty)\times[0, \infty)$ and nondecreasing in r.

We note that $g(t)$ is a general deviating argument, that is, it is allowed to be advanced ($g(t)\geqq t$) or retarded ($g(t)\leqq t$) or otherwise.

Equation (A) is called *superlinear* or *sublinear* according to whether $\omega(t, r)/r$ is nondecreasing or nonincreasing in r for $r>0$.

In what follows we restrict our attention to solutions $x(t)$ of (A) which exist on some half-line $[T_x, \infty)$ and are nontrivial on any infinite subintervals of $[T_x, \infty)$. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros; otherwise it is said to be *nonoscillatory*.

Our purpose here is to study the asymptotic behavior of solutions of equation (A), attempting to determine the rate of growth or decay of nonoscillatory solutions as well as of oscillatory solutions. Our results extend some of the results of [2] for the second order case.

It would be of interest to observe that there is a kind of duality between the results of this paper and those of the paper [3] in which equations of the form (A) satisfying the condition

$$\int_a^\infty \frac{dt}{p_i(t)}=\infty, \qquad 1\leqq i\leqq n-1,$$

are discussed.

2. Possible behavior of all solutions. We need the following notation:

$$\phi_{i,i}(t,s)=1, \qquad 0\leqq i\leqq n-1,$$
$$\phi_{i,j}(t,s)=\int_s^t \frac{ds_{i+1}}{p_{i+1}(s_{i+1})}\int_s^{s_{i+1}} \frac{ds_{i+2}}{p_{i+2}(s_{i+2})}\int_s^{s_{i+2}}\cdots\int_s^{s_{j-1}}\frac{ds_j}{p_j(s_j)},$$
$$0\leqq i<j\leqq n-1,$$
$$\pi_i(t)=|\phi_{0,i}(t,\infty)|, \quad \rho_i(t)=\phi_{i,n-1}(\infty,t), \qquad 0\leqq i\leqq n-1,$$
$$\alpha_i(t)=\frac{\pi_i(t)}{\pi_{i-1}(t)}, \quad \beta_i(t)=\max\left\{1, \frac{\pi_{i-1}(t)\rho_{i-1}(t)}{\pi_i(t)\rho_i(t)}\right\}, \qquad 1\leqq i\leqq n-1,$$
$$g^*(t)=\max\{g(t),t\}, \qquad g_*(t)=\min\{g(t),t\},$$
$$h^*(t)=\sup_{a\leqq s\leqq t} g^*(s), \qquad h_*(t)=\inf_{s\geqq t} g_*(s).$$

The possible behavior of all solutions of (A) is described in the following theorem.

Theorem 1. *Suppose that either* (A) *is superlinear and*

$$(1) \qquad \int^\infty \alpha_i(g_*(t))\beta_i(t)\rho_i(t)\omega(t, c\pi_{i-1}(g(t)))dt<\infty$$

for all $c>0$ *and* i *with* $1\leqq i\leqq n-1$, *or* (A) *is sublinear and*

$$(2) \qquad \int^\infty \frac{\alpha_i(g_*(t))}{\alpha_i(g(t))}\beta_i(t)\rho_i(t)\omega(t, c\pi_i(g(t)))dt<\infty$$

for all $c>0$ *and* i *with* $1\leqq i\leqq n-1$.

If $x(t)$ *is a solution of* (A), *then one of the following cases holds:*

(I) $\limsup_{t\to\infty}|x(t)|=\infty$;

(II) *There exists an integer* $k, 0\leqq k\leqq n-1$, *and a nonzero number* c_k *such that*

$$\lim_{t\to\infty}\frac{x(t)}{\pi_k(t)}=c_k;$$

(III) $\lim_{t\to\infty}\frac{x(t)}{\pi_{n-1}(t)}=0.$

A key to the proof of Theorem 1 is the following

Lemma 1. *Let* $1\leqq k\leqq n-1$ *and suppose that*

$$\int^\infty \alpha_k(g_*(t))\beta_k(t)\rho_k(t)\omega(t, \pi_{k-1}(g(t)))dt<\infty$$

if (A) *is superlinear and that*

$$\int^\infty \frac{\alpha_k(g_*(t))}{\alpha_k(g(t))}\beta_k(t)\rho_k(t)\omega(t, \pi_k(g(t)))dt<\infty$$

if (A) *is sublinear.*

If $x(t)$ *is a solution of* (A) *such that* $x(t)=o(\pi_{k-1}(t))$ *as* $t\to\infty$, *then* $x(t)=O(\pi_k(t))$ *as* $t\to\infty$.

We say that condition (G^*) [resp. (G_*)] is satisfied if there is a sequence $\{t_\nu\}_{\nu=1}^\infty$ such that $t_\nu\to\infty$ as $\nu\to\infty$ and $h^*(t_\nu)=t_\nu$ [resp. $h_*(t_\nu)=t_\nu$] for $\nu=1,2,\cdots$.

With the aid of these additional conditions on $g(t)$ it is possible to rule out Case (I) and/or Case (III) from the possibilities listed in

Theorem 1.

Lemma 2. *Suppose that* (A) *is superlinear and* (G_*) *is satisfied. If*

$$\int^{\infty} \omega(t, \pi_{n-1}(g(t)))dt < \infty,$$

then every solution $x(t)$ *of* (A) *satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{\pi_{n-1}(t)} > 0.$$

Lemma 3. *Suppose that* (A) *is sublinear and* (G^*) *is satisfied. If*

$$\int^{\infty} \rho_0(t)\omega(t, 1)dt < \infty,$$

then every solution $x(t)$ *of* (A) *is bounded.*

3. Behavior of nonoscillatory solutions. As easily verified, $\{\pi_k(t) : 0 \leqq k \leqq n-1\}$ is a fundamental set of solutions of $L_n x(t)=0$ on $[a, \infty)$. A sufficient condition for (A) to have a nonoscillatory solution such that $\lim_{t\to\infty} x(t)/\pi_k(t) = \text{const.} \neq 0$ is given in the following theorem which is an extension of a result of Granata [1].

Theorem 2. *Let* $0 \leqq k \leqq n-1$ *and suppose that*

$$\int^{\infty} \rho_k(t)\omega(t, c\pi_k(g(t)))dt < \infty \quad \textit{for some } c>0. \tag{3}$$

Then (A) *has solutions* $x(t)$, $y(t)$ *such that*

$$\lim_{t\to\infty} \frac{x(t)}{\pi_k(t)} = \frac{c}{2}, \qquad \lim_{t\to\infty} \frac{y(t)}{\pi_k(t)} = -\frac{c}{2}.$$

For example, the required solution $x(t)$ is obtained as a solution of the integral equation

$$x(t) = \frac{c}{2}\pi_k(t) + \Phi_k x(t),$$

where

$$\Phi_k x(t) = \int_t^{\infty} \phi_{0,n-1}(t, s) f(s, x(g(s)))ds \qquad \text{if } k=n-1,$$

$$\Phi_k x(t) = \int_t^{\infty} \frac{\phi_{0,k}(t, s)}{p_{k+1}(s)} \int_T^s \phi_{k+1,n-1}(s, \sigma) f(\sigma, x(g(\sigma)))d\sigma ds \qquad \text{if } 0 \leqq k \leqq n-2,$$

and T is a sufficiently large constant.

Suppose that (A) is either superlinear or sublinear. Then the assumptions of Theorem 1 implies that (3) holds for all $c>0$ and k with $0 \leqq k \leqq n-1$. Therefore, under the assumptions of Theorem 1 equation (A) actually possesses for each k, $0 \leqq k \leqq n-1$, nonoscillatory solutions which are asymptotic to $\pi_k(t)$ as $t \to \infty$.

It can be shown that in certain cases all nonoscillatory solutions of (A) fall into the class (II) of Theorem 1, that is, they behave like the solutions of the equation $L_n x(t)=0$.

Theorem 3. *Let* n *be even and* $xf(t, x) \geqq 0$ *on* $[a, \infty) \times (-\infty, \infty)$.

Suppose that the hypotheses of Theorem 1 *are satisfied.*

Then, any nonoscillatory solution of (A) *behaves as in Case* (II) *of Theorem* 1.

Theorem 4. *Let n be odd. Suppose that*

(i) (A) *is superlinear,* $xf(t,x)\geqq 0$ *on* $[a,\infty)\times(-\infty,\infty)$, (G_*) *is satisfied and* (1) *holds for all* $c>0$ *and every* i, $1\leqq i\leqq n-1$, *or*

(ii) (A) *is sublinear,* $xf(t,x)\leqq 0$ *on* $[a,\infty)\times(-\infty,\infty)$, (G^*) *is satisfied and* (2) *holds for all* $c>0$ *and every* i, $1\leqq i\leqq n-1$.

Then, any nonoscillatory solution of (A) *behaves as in Case* (II) *of Theorem* 1.

4. Behavior of oscillatory solutions. With regard to the asymptotic behavior of oscillatory solutions of (A) we have the following information.

Theorem 5. (i) *Assume that* (A) *is superlinear and* (G_*) *is satisfied. If* (1) *holds for all* $c>0$ *and every* i, $1\leqq i\leqq n-1$, *then every oscillatory solution* $x(t)$ *of* (A) *has the property*

$$\limsup_{t\to\infty}|x(t)|=\infty.$$

(ii) *Assume that* (A) *is sublinear and* (G^*) *is satisfied. If* (2) *holds for all* $c>0$ *and every* i, $1\leqq i\leqq n-1$, *then every oscillatory solution of* (A) *has the property*

$$\lim_{t\to\infty}\frac{x(t)}{\pi_{n-1}(t)}=0.$$

As a consequence of Theorem 5 we have the following nonoscillation theorem for *almost linear* equations of the form (A).

Theorem 6. *Suppose that*

$$|f(t,x)|\leqq q(t)|x| \qquad \text{for } (t,x)\in[a,\infty)\times(-\infty,\infty),$$

where $q(t)$ *is continuous and positive on* $[a,\infty)$. *Suppose in addition that both* (G^*) *and* (G_*) *are satisfied. If*

$$\int^{\infty}\alpha_i(g_*(t))\beta_i(t)\pi_{i-1}(g(t))\rho_i(t)q(t)dt<\infty \qquad \text{for } 1\leqq i\leqq n-1,$$

then all solutions of (A) *are nonoscillatory.*

References

[1] A. Granata: Singular Cauchy problems and asymptotic behavior for a class of n-th order differential equations. Funkcial. Ekvac. (to appear).

[2] Y. Kitamura and T. Kusano: Asymptotic properties of solutions of two-dimensional differential systems with deviating argument. Hiroshima Math. J. (to appear).

[3] Y. Kitamura, T. Kusano, and M. Naito: Asymptotic properties of solutions of n-th order differential equations with deviating argument. Proc. Japan Acad., **54**A, 13–16 (1978).