39. Note on Quasi-Domination in the Sense of K. Borsuk

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In this paper we shall prove that if an approximatively 1-connected pointed continuum (X, x_0) is pointed quasi-dominated by a pointed FANR, X has the shape of a compact polyhedron. We shall give an example that the compactness of Y in the theorem is essential. Throughout this paper all maps are continuous and by a space we mean a topological space. We mean by $\mathcal{W}$ the category of spaces having the homotopy type of CW-complexes and homotopy classes of maps.

Let $\underline{X}=\{X_\alpha, [p_{\alpha\alpha'}], A\}$ and $\underline{Y}=\{Y_\beta, [q_{\beta\beta'}], B\}$ be objects of pro-$\mathcal{W}$, where $[f]$ denotes the homotopy class of the map f. We say that X is *quasi-dominated* by $\underline{Y}$ (notation: $\underline{X}\overset{q}{\leq}\underline{Y}$) if for any $\alpha_0 \in A$ there exist two system maps $\underline{f}=\{f, [f_\beta], B\}: \underline{X}\to\underline{Y}$ and $\underline{g}=\{g, [g_\alpha], A\}: \underline{Y}\to\underline{X}$ such that there exists $\alpha_1 \in A$ such that $\alpha_1 \geq \alpha_0$, $fg(\alpha_0)$ and $g_{\alpha_0} f_{g(\alpha_0)} p_{fg(\alpha_0)\alpha_1} \simeq p_{\alpha_0\alpha_1}$. Let X and Y be spaces. We say that X is *quasi-dominated* by Y (notation: $X\overset{q}{\leq}Y$) if there exist $\underline{X}=\{X_\alpha, [p_{\alpha\alpha'}], A\}$ and $\underline{Y}=\{Y_\beta, [q_{\beta\beta'}], B\}$ of objects of pro-$\mathcal{W}$ such that $\underline{X}$ and $\underline{Y}$ are associated with X and Y respectively (see [9]) and $\underline{X}\overset{q}{\leq}\underline{Y}$.

It is clear that the definition of quasi-domination of spaces is independent of chosing objects of pro-$\mathcal{W}$ associated with X and Y. We can easily prove that for compacta our definition is equivalent to the definition of K. Borsuk [2] (cf. [7]).

Analogously the notation of pointed quasi-domination of pointed spaces (notation: $(X, x_0)\overset{q}{\leq}(Y, y_0)$) is defined.

The following is easy to prove.

Theorem 1. *Let X and Y be spaces. If $X\overset{q}{\leq}Y$ and the shape of Y is trivial, the shape of X is trivial.*

Thus, if a compactum X is quasi-dominated by an FAR-space Y, then X is also an FAR-space (see [1]).

From Theorem 1 the following problem is raised: *if X is a compactum quasi-dominated by an* FANR-*space Y, is X an* FANR-*space?*

We obtain the following partial answers.

Theorem 2. *Let (X, x_0) and (Y, y_0) be pointed continua. Suppose that X is approximatively* 1-*connected* (*see* [1]) *and (Y, y_0) is a pointed* FANR-*space. If $(X, x_0)\overset{q}{\leq}(Y, y_0)$, then X has the shape of a compact* *polyhedron.*

Proof. Since $(X, x_0) \overset{q}{\leq} (Y, y_0)$ and (Y, y_0) is a pointed FANR-space, $(X, x_0) \overset{q}{\leq} (P, p_0)$ for some pointed connected compact polyhedron (P, p_0). Let $\{(X_n, x_n), p_{n,n+1}, N\}$ be an inverse sequence of compact connected polyhedra such that $\varprojlim \{(X_n, x_n), p_{n,n+1}, N\} = (X, x_0)$. Since $(X, x_0) \overset{q}{\leq} (P, p_0)$, there are an increasing sequence $1 = i(1) < i(2) < \cdots$ of integers and sequences of maps $f_n : (P, p_0) \to (X_{i(n)}, x_{i(n)})$, $g_n : (X_{i(n)}, x_{i(n)}) \to (P, p_0)$ $(n=1, 2, \cdots)$ such that $f_n g_{n+1} \simeq p_{i(n), i(n+1)}$ rel. $x_{i(n+1)}$. For each $n=1, 2, \cdots$, set

$$(X'_n, x'_n) = (P, p_0), \ q_{n,n+1} = g_n f_n : (X'_{n+1}, x'_{n+1}) \to (X'_n, x'_n)$$

and

$$(X', x'_0) = \varprojlim \{(X'_n, x'_n), q_{n,n+1}, N\}.$$

Then $\mathrm{Sh}(X', x'_0) = \mathrm{Sh}(X, x_0)$. Hence by S. Nowak [10] there are a simply connected pointed polyhedron (Q, q_0) and an inverse sequence $\{(Z_n, z_n), r_{n,n+1}, N\}$ such that $(Z_n, z_n) = (Q, q_0)$ for each $n=1, 2, \cdots$ and $\varprojlim \{(Z_n, z_n), r_{n,n+1}, N\} = (X', x'_0)$. Since for every $k=1, 2, \cdots$ $\pi_k(Q, q_0)$ is a finitely generated abelian group (see [11, Corollary 9.6.16]), $\pi_k(X', x'_0) \cong \varprojlim \{\pi_k(Z_n, z_n), \pi_k(r_{n,n+1}), N\}$ is isomorphic to a subgroup of $\pi_k(Q, q_0)$ for every $k=1, 2, \cdots$ by A. Kadlof [6]. Hence $\pi_k(X', x'_0)$ is countable for every $k=1, 2, \cdots$. Moreover it is clear that (X', x'_0) is pointed movable (cf. [2] and [8]) and $\dim X' \leq \dim Q < +\infty$. Hence by T. Watanabe [12] (or J. Dydak [3]) (X', x'_0) is a pointed FANR-space. Since (X, x_0) is approximatively 1-connected, the Wall obstruction $w(\check{\pi}_1(X, x_0)) \in \tilde{K}^0(\check{\pi}_1(X, x_0))$ vanishes. Therefore (X, x_0) has the shape of a pointed compact polyhedron by D. Edwards and R. Geoghegan [4]. The proof is complete.

Theorem 3. *Let (X, x_0) be a pointed continuum. Let (P, p_0) is a compact connected polyhedron such that $\pi_1(P, p_0)$ is finite. If $(X, x_0) \overset{q}{\leq} (P, p_0)$, then* $\mathrm{Sh}(X, x_0) \leq \mathrm{Sh}(P, p_0)$ *and hence (X, x_0) is a pointed* FANR-*space.*

Proof. By the same way as the proof of Theorem 2, there is an inverse sequence $\{(X'_n, x'_n), q_{n,n+1}, N\}$ such that for each $n=1, 2, \cdots$ $(X'_n, x'_n) = (P, p_0)$ and $(X', x'_0) = \varprojlim \{(X'_n, x'_0), q_{n,n+1}, N\}$ has the shape of (X, x_0). Since (X', x'_0) is P-like movable continuum, $\mathrm{Sh}(X', x'_0) \leq \mathrm{Sh}(P, p_0)$ by A. Kadlof [6]. Therefore $\mathrm{Sh}(X, x_0) \leq \mathrm{Sh}(P, p_0)$.

Remark. In general even if X and Y are compacta, $X \overset{q}{\leq} Y$ does not imply $\mathrm{Sh}(X) \leq \mathrm{Sh}(Y)$ (see [2]). But under the condition of Theorem 2 $(X, x_0) \overset{q}{\leq} (P, p_0)$ is equivalent to $\mathrm{Sh}(X, x_0) \leq \mathrm{Sh}(P, p_0)$.

Finally we give an example in which (X, x_0) is an approximatively 1-connected movable continuum which is not an FANR-space and (Y, y_0) is a finite dimensional locally compact metric ANR-space such that

$(X, x_0) \overset{q}{\leq} (Y, y_0)$.

Example. For every $n=1, 2, \cdots$ let

$$X_n = \left\{(x, y, z) \in R^3 \middle| \left\{x - \frac{2n+1}{2n(n+1)}\right\}^2 + y^2 + z^2 = \left\{\frac{1}{2n(n+1)}\right\}^2\right\}$$

and

$$Y_n = \{(x, y, z) \in R^3 \mid (x-2n+1)^2 + y^2 + z^2 = 1\}.$$

Put

$$X = \{(0, 0, 0)\} \cup \left(\bigcup_{n=1}^{\infty} X_n\right) \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n.$$

It is clear that X is an approximatively 1-connected, movable continuum but not an FANR and Y is a 2-dimensional, locally compact metric ANR-space. Moreover since for every $k=1, 2, \cdots$ the set $\bigcup_{n=1}^{k} X_n$ is homeomorphic to a retract of Y, we have $X \overset{q}{\leq} Y$. Since X is approximatively 1-connected $(X, x_0) \overset{q}{\leq} (Y, y_0)$ for any points $x_0 \in X$ and $y_0 \in Y$.

Note that $\check{H}_2(X) \cong Z \oplus Z \oplus \cdots$ and $\check{H}_2(Y) \cong Z \times Z \times \cdots$, where $\check{H}_*$ is the Čech homology. Thus $\check{H}_2(X)$ is not a direct summand of $\check{H}_2(Y)$ (see [5, p. 163]). The following problem is open (cf. K. Borsuk [2, Problem (10.5)]).

Problem. *If X and Y are compacta such that $X \overset{q}{\leq} Y$, then is $\check{H}_n(X : G)$ a direct summand of $\check{H}_n(Y : G)$ for every $n=1, 2, \cdots$ and every abelian group G?*

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References

[1] K. Borsuk: Theory of Shape. PWN, Warszawa (1975).

[2] ——: Some quantitative properties of shapes. Fund. Math., **93**, 197–212 (1976).

[3] J. Dydak: An algebraic condition characterizing FANR-spaces. Bull. Acad. Pol., **24**, 501–503 (1976).

[4] D. Edwards and R. Geoghegan: Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction. Ann. of Math., **10**, 521–535 (1975).

[5] L. Fuchs: Infinite Abelian Groups II. Academic Press (1973).

[6] A. Kadlof: On some properties of P-like compacta. Bull. Acad. Pol., **25**, 63–66 (1977).

[7] S. Mardesić and J. Segal: Equivalence of the Borsuk and the ANR-systems approach to shape. Fund. Math., **72**, 61–68 (1971).

[8] ——: Movable compacta and ANR-systems. Bull. Acad. Pol., **18**, 649–654 (1970).

[9] K. Morita: On shapes of topological spaces. Fund. Math., **86**, 251–259 (1975).

[10] S. Nowak: On the fundamental dimension of approximatively 1-connected compacta. Fund. Math., **89**, 61–78 (1975).

[11] E. Spanier: Algebraic Topology. McGraw-Hill (1966).
[12] T. Watanabe: On Čech homology and stability theorem in shape theory. J. Math. Soc. Japan, **29**, 655–664 (1977).