48. On the Normal Generation by a Line Bundle on an Abelian Variety

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(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1978)

Let k be an algebraically closed field of characteristic $p \geqq 0$, X an abelian variety over k, and L an ample invertible sheaf on X. In the previous paper [7], the author proved, unfortunately providing $p \neq 2, 3$, that the embedded variety of X into the projective space $\boldsymbol{P}(\Gamma(L^3))$ by means of the global sections of L^3, is ideal-theoretically an intersection of cubics. But he has recently found that the method in it can be extended for $p=2$. Namely, it can be checked easily even for $p=2$ that the canonical map $\Gamma(L^2 \otimes P_\alpha) \otimes \Gamma(L^2 \otimes P_{-\alpha}) \rightarrow \Gamma(L^4)$ is surjective for almost all α in $\hat{X}$, and which was the only obstacle to the proof of the fact for $p=2$ in [7]. Moreover, as mentioned in [7], the surjectivity of the canonical maps lead us easily to the main results given in [1], [5] and [6].

We start with the following Mumford's theta structure theorem.

Theorem (Mumford's theta structure theorem). *Let M be a non-degenerate invertible sheaf on X of index i, and*

$$1 \longrightarrow \boldsymbol{G}_m \longrightarrow \mathcal{G}(M) \xrightarrow{j(M)} K(M) \longrightarrow 0$$

the theta group scheme of M, where $K(M)$ is the scheme-theoretic kernel of the homomorphism $\phi_M : X \rightarrow \hat{X}$ defined by $x \mapsto T_x^* M \otimes M^{-1}$. Then the canonical action U of $\mathcal{G}(M)$ on $H^i(M) = H^i(X, M)$ is the unique irreducible representation of $\mathcal{G}(M)$, with $\boldsymbol{G}_m$ acting naturally (cf. Appendix to [6]).

Corollary. *Under the same notation as in above, let V and W be two subspaces of $H^i(M)$ with $V \supset W \neq \{0\}$. Assume that for any local ring $(B, \mathfrak{M})$ over k with the residue field k and any B-valued point λ of $\mathcal{G}(M)$, $U_\lambda(W \otimes B) \subset V \otimes B$. Then we have $V = H^i(M)$.*

Proof. Let $R = \Gamma(\mathcal{G}(M), \mathcal{O}_{\mathcal{G}(M)})$, and $\sigma : H^i(M) \rightarrow H^i(M) \otimes R$ be the co-module structure corresponding to the action U. We denote by γ the composition:

$$H^i(M) \otimes R^* \xrightarrow{\sigma \otimes 1_{R^*}} H^i(M) \otimes R \otimes R^* \xrightarrow{1_{H^i(M)} \otimes \langle\,\rangle} H^i(M),$$

where $R^* = \mathrm{Hom}_k(R, k)$ and $\langle\ \rangle$ stands for "contraction". Here we put $\overline{W} = \gamma(W \otimes R^*)$. Then obviously $\{o\} \neq W \subset \overline{W} \subset V$ and $\overline{W}$ becomes a stable subspace under the action U. Hence, by Mumford's theta structure theorem, we have $\overline{W} = V = H^i(M)$. Q.E.D.

The following is the main result in this paper, and which is a generalization of Proposition 1.5 in [7] for any characteristic.

Main theorem. *Let L be any ample invertible sheaf on X. Then for any α, β in $\hat{X}$, if we take a point γ of $\hat{X}$ in general position,*

$$\Gamma(L^2 \otimes P_{\alpha+\gamma}) \otimes \Gamma(L^2 \otimes P_{\beta-\gamma}) \to \Gamma(L^4 \otimes P_{\alpha+\beta})$$

is surjective, where P is the Poincaré invertible sheaf on $X \times \hat{X}$ and $P_{\hat{x}} = P|_{X \times \{\hat{x}\}}$ for any point $\hat{x} \in \hat{X}$.

Proof. If necessary, slightly modifying α and β, we may assume that L is symmetric. Let $\xi : X \times X \to X \times X$ be the homomorphism defined by $(x, y) \mapsto (x-y, x+y)$. Then we have an isomorphism

$$\xi^*(p_1^*(L^2 \otimes P_\alpha) \otimes p_2^*(L^2 \otimes P_\beta)) \xrightarrow{\sim} p_1^*(L^4 \otimes P_{\alpha+\beta}) \otimes p_2^*(L^4 \otimes P_{\beta-\alpha}),$$

where $p_i : X \times X \to X$ is the projection to the i-th component for each $i = 1, 2$. This isomorphism defines a lifting of the group $K = \ker(\xi)$:

$$\begin{array}{ccccccc} 1 \longrightarrow G_m \longrightarrow & \mathcal{G}(p_1^*(L^4 \otimes P_{\alpha+\beta}) \otimes p_2^*(L^4 \otimes P_{\beta-\alpha})) & \xrightarrow{j} & K(L^4) \times K(L^4) \longrightarrow 0, \\ & \cup & & \cup \\ & K^* & \xrightarrow{\sim} & K \end{array}$$

where $j = j(p_1^*(L^4 \otimes P_{\alpha+\beta}) \otimes p_2^*(L^4 \otimes P_{\beta-\alpha}))$. Moreover, by the descent theory, passing through ξ^*, $\Gamma(L^2 \otimes P_\alpha) \otimes \Gamma(L^2 \otimes P_\beta)$ is isomorphic to the K^*-invariant subspace $(\Gamma(L^4 \otimes P_{\alpha+\beta}) \otimes \Gamma(L^4 \otimes P_{\beta-\alpha}))^{K^*}$ of $\Gamma(L^4 \otimes P_{\alpha+\beta}) \otimes \Gamma(L^4 \otimes P_{\beta-\alpha})$. If we denote by $\mathcal{G}^*$ the centralizer of K^*, then we have a canonical exact sequence:

$$1 \longrightarrow K^* \longrightarrow \mathcal{G}^* \xrightarrow{\mathcal{G}(\xi)} \mathcal{G}(p_1^*(L^2 \otimes P_\alpha) \otimes p_2^*(L^2 \otimes P_\beta)) \longrightarrow 0,$$

and a commutative diagram

$$\begin{array}{ll} & (\Gamma(L^2 \otimes P_\alpha) \otimes B) \otimes_B (\Gamma(L^2 \otimes P_\beta) \otimes B) \\ & \qquad \downarrow U_{\mathcal{G}(\xi)(\mu)} \\ (1) & (\Gamma(L^2 \otimes P_\alpha) \otimes B) \otimes_B (\Gamma(L^2 \otimes P_\beta) \otimes B) \\ & \qquad \xrightarrow{B \otimes \xi^*} (\Gamma(L^4 \otimes P_{\alpha+\beta}) \otimes B) \otimes_B (\Gamma(L^4 \otimes P_{\beta-\alpha}) \otimes B) \\ & \qquad\qquad\qquad\qquad \downarrow U_\mu \\ & \qquad \xrightarrow{B \otimes \xi^*} (\Gamma(L^4 \otimes P_{\alpha+\beta}) \otimes B) \otimes_B (\Gamma(L^4 \otimes P_{\beta-\alpha}) \otimes B) \end{array}$$

for any k-algebra B and any B-valued point μ of $\mathcal{G}(p_1^*(L^4 \otimes P_{\alpha+\beta}) \otimes p_2^*(L^4 \otimes P_{\beta-\alpha}))$. For any integer n, we put $X_n = \{x \in X \mid nx = 0\}$. Since $X_2 \times X_2$ is isotropic in $K(p_1^*(L^4 \otimes P_{\alpha+\beta}) \otimes p_2^*(L^4 \otimes P_{\beta-\alpha}))$ and contains K, we can lift $X_2 \times X_2$ up to a level subgroup $(X_2 \times X_2)^*$ containing K^*. Moreover, obviously there exist level subgroups $X_2^* \subset \mathcal{G}(L^4 \otimes P_{\alpha+\beta})$ and $X_2^{**} \subset \mathcal{G}(L^4 \otimes P_{\beta-\alpha})$ such that $X_2^* \times X_2^{**}$ is isomorphic to $(X_2 \times X_2)^*$ passing through the canonical homomorphism $\pi : \mathcal{G}(L^4 \otimes P_{\alpha+\beta}) \times \mathcal{G}(L^4 \otimes P_{\beta-\alpha}) \to \mathcal{G}(p_1^*(L^4 \otimes P_{\alpha+\beta}) \otimes p_2^*(L^4 \otimes P_{\beta-\alpha}))$. Here we take non-trivial sections $\theta \in \Gamma(L^4 \otimes P_{\alpha+\beta})^{X_2^*}$ and $\theta' \in \Gamma(L^4 \otimes P_{\beta-\alpha})^{X_2^{**}}$. Then, since $X_2^* \times X_2^{**} \simeq (X_2 \times X_2)^* \supset K^*$, $\theta \otimes \theta' \in (\Gamma(L^4 \otimes P_{\alpha+\beta}) \otimes \Gamma(L^4 \otimes P_{\beta-\alpha}))^{K^*}$, i.e., there exists an element Θ in $\Gamma(L^2 \otimes P_\alpha) \otimes \Gamma(L^2 \otimes P_\beta)$ such that $\xi^* \Theta = \theta \otimes \theta'$. Now we choose a point $y_0 \in X$ satisfying the condition.

(2) $\quad \theta'(y_0+x)\neq 0 \quad$ for any k-valued point x of $K(L^4\otimes P_{\beta-\alpha})$,
and we put $\gamma=-2\phi_L(y_0)$. Then, since the diagram

$$\begin{tikzcd}
X\times X \arrow[d, "T_{-y_0}\times T_{y_0}"'] & X \simeq X\times \operatorname{Spec}(k(y_0)) \arrow[l, "\Delta"'] \arrow[d, "1_X\times y_0"] \\
X\times X & X\times X \arrow[l, "\xi"']
\end{tikzcd}$$

commutes, we have a commutative diagram

$$(3)\qquad \begin{tikzcd}
\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma}) \arrow[r, "\Delta^*"] \arrow[d, phantom, "\wr"] & \Gamma(L^4\otimes P_{\alpha+\beta}) \\
\Gamma(T^*_{-y_0}L^2\otimes P_\alpha)\otimes\Gamma(T^*_{y_0}L^2\otimes P_\beta) & \\
\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_\beta) \arrow[u, "T^*_{y_0}\otimes T^*_{y_0}"] \arrow[r, "\xi^*"] & \Gamma(L^4\otimes P_{\alpha+\beta})\otimes\Gamma(L^4\otimes P_{\beta-\alpha}). \arrow[uu, "1^*_X\otimes y^*_0"']
\end{tikzcd}$$

Therefore, if we put $V=\operatorname{Im}[\tau=\Delta^*:\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma})\to\Gamma(L^4\otimes P_{\alpha+\beta})]$, it contains the subspace W spanned by only one element θ. Hence, by virtue of Corollary to Mumford's theta structure theorem, we have only to show that

$$\begin{cases}\text{for any local ring } (B,\mathfrak{M}) \text{ over } k \text{ with the residue field } k \\ \text{and any } B\text{-valued point } \lambda \text{ of } \mathcal{G}(L^4\otimes P_{\alpha+\beta}),\ U_\lambda(W\otimes B)\subset V\otimes B.\end{cases}$$

So, let $(B,\mathfrak{M})$ be such a local ring, and λ be such a point of $\mathcal{G}(L^4\otimes P_{\alpha+\beta})$. We choose a B-valued point λ' of $\mathcal{G}(L^4\otimes P_{\beta-\alpha})$ such that $j(L^4\otimes P_{\beta-\alpha})(\lambda')=j(L^4\otimes P_{\alpha+\beta})(\lambda)$. Since $\Delta(K(L^4))\subset K(p_1^*(L^4\otimes P_{\alpha+\beta})\otimes p_2^*(L^4\otimes P_{\beta-\alpha}))$ and $j^{-1}(\Delta(K(L^4)))\subset\mathcal{G}^*$, $\pi(\lambda,\lambda')\in\mathcal{G}^*$. Therefore, by commutativity of (1), we have

$$U_\lambda\theta\otimes U_{\lambda'}\theta'=(\xi^*\otimes B)(U_{G(\xi)\pi(\lambda,\lambda')}\Theta).$$

Here we put $S=\operatorname{Spec}(B)$, $\iota:\operatorname{Spec}(B/\mathfrak{M})\to S$ the closed point, $X_S=X\times S$, $L_S=L\otimes B$ and $y'_0=y_0\times 1_S:\operatorname{Spec}(k)\times S=S\to X\times S$. Then we have commutative diagrams

$$\begin{tikzcd}
X_S\times_S X_S\simeq X\times X\times S \arrow[d, "T_{-y'_0}\times T_{y'_0}"'] & X_S\simeq X_S\times_S S \arrow[l, "\Delta\times 1_S"'] \arrow[d, "1_{X_S}\times y'_0"] \\
X_S\times_S X_S\simeq X\times X\times S & X\times X\times S\simeq X_S\times_S X_S \arrow[l, "\xi\times 1_S"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\{\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes B\}\otimes_B\{\Gamma(L^2\otimes P_{\beta-\gamma})\otimes B\}\simeq\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma})\otimes B \arrow[d, phantom, "\wr"] \\
\Gamma(T^*_{-y'_0}(L^2\otimes P_\alpha)_S)\otimes\Gamma(T^*_{y'_0}(L^2\otimes P_\beta)_S) \\
\{\Gamma(L^2\otimes P_\alpha)\otimes B\}\otimes_B\{\Gamma(L^2\otimes P_\beta)\otimes B\}\simeq\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_\beta)\otimes B \arrow[u, "T^*_{-y'_0}\otimes T^*_{y'_0}"]
\end{tikzcd}$$

$$\xrightarrow{\tau\otimes B}\Gamma(L^4\otimes P_{\alpha+\beta})\otimes B$$

$$\Big\uparrow (1_{X_S}\times y'_0)^*$$

$$\xrightarrow{\xi^*\otimes B}\{\Gamma(L^4\otimes P_{\alpha+\beta})\otimes B\}\otimes_B\{\Gamma(L^4\otimes P_{\beta-\alpha})\otimes B\}.$$

Hence

$$U_\lambda\theta\cdot(U_{\lambda'}\theta')(y_0')\in \mathrm{Im}\,(\tau\otimes B)=V\otimes B.$$

On the other hand, if we put $j(L^4\otimes P_{\beta-\alpha})(\lambda')=u\in K(L^4\otimes P_{\beta-\alpha})(B)$,

$$\begin{aligned}\iota^*((U_{\lambda'}\theta')(y_0'))&=\iota^*(u+y_0')^*\theta'\\&=\theta'(y_0+u\circ\iota).\end{aligned}$$

Since $u\circ\iota\in K(L^4\otimes P_{\beta-\alpha})$, the condition (2) implies $\theta'(y_0+u\circ\iota)\neq 0$, i.e., $(U_\lambda\theta')(y_0')$ is a unit of B. Therefore $U_\lambda\theta\in V\otimes B$ and we are done.

Q.E.D.

References

[1] S. Koizumi: Theta relations and projective normality of abelian varieties. Amer. J. Math., **98**(3), 865–889 (1976).

[2] ——: The rank theorem on matrices of theta functions. J. Fac. Sci. University of Tokyo, Sec. IA, **24**(1), 115–122 (1977).

[3] D. Mumford: On the equations defining abelian varieties. I. Invent. Math., **1**, 287–354 (1966).

[4] ——: Varieties defined by quadratic equations. Questioni sulle Varieta Algebriche, Corsi dal C.I.M.E., Edizioni Cremonese, Roma (1969).

[5] T. Sekiguchi: On projective normality of abelian varieties. J. Math. Soc. Japan, **28**(2), 307–322 (1976).

[6] ——: On projective normality of abelian varieties. II. Ibid., **29**(4), 709–727 (1977).

[7] ——: On the cubics defining abelian varieties (to appear in J. Math. Soc. Japan, **30**(4) (1978)).