10. *On Weak Evolution Operators with Constant Coefficients and Spreading of Wave Packets*

By Kimimasa NISHIWADA*)
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1979)

Some algebraic properties of a polynomial $P(\xi)$ are often determined if the corresponding partial differential equation $P(D)u=0$ has a nontrivial solution of special type. One of the important such results is the theorem due to F. John [4]. In that article he discussed the relation between the weak hyperbolicity of polynomials and the existence of solutions that represent the propagation of support with finite speed. L. Ehrenpreis, in his book [1, chap. IX], treated a similar problem even for non-kowalevskian operators.

In this note we shall state that we can generalize their results (at least in the single equation case), introducing the notions of weak evolution operators and wave packet spreading. These notions, in a way, extend those of weak hyperbolicity and support propagation.

1. Weak evolution operators. Let $P(\sigma, \xi)$ be a polynomial in $n+1$ variables $(\sigma, \xi)=(\sigma, \xi_1, \cdots, \xi_n)$ with complex coefficients of the form

$$P(\sigma, \xi)=\sigma^l+\sum_{j=1}^{l} a_j(\xi)\sigma^{l-j}, \tag{1}$$

for some integer $l\geqq 1$. Let us put $p=\operatorname*{Max}_j (\deg a_j)/j$ and denote the homogeneous part of degree pj of a_j by a_j^0. Then the principal part of P is defined by

$$P^0(\sigma, \xi)=\sigma^l+\sum_{j=1}^{l} a_j^0(\xi)\sigma^{l-j}. \tag{2}$$

As usual we write $(D_t, D_x)=\dfrac{1}{i}\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n}\right)$ and define the partial differential operator $P(D_t, D_x)$ in $\boldsymbol{R}^{n+1}$. We owe the following definition, as well as the above one of P^0, to S. Mizohata [6].

Definition 1. Given a rational number $\alpha>0$, we shall call $P(D_t, D_x)$ the weak α-evolution operator with respect to the half space $H=\{(t, x);\ t\geqq 0, x\in \boldsymbol{R}^n\}$ if either $\alpha>p$ or the following holds; $\alpha=p$ is an even integer and the imaginary parts of all the roots of $P^0(\sigma, \xi)=0$, $\xi\in\boldsymbol{R}^n$, are $\geqq 0$, or $\alpha=p$ is an odd integer and all those roots are real. We shall call P simply the weak α-evolution operator (resp. of S-type) if either $\alpha>p$, or $\alpha=p$ is an integer and real are all the roots of $P^0(\sigma, \xi)$

*) Partially supported by the Sakkokai Foundation.

$=0$, $\xi \in \boldsymbol{R}^n$, (resp. of $P^0(\sigma, \theta_1\xi_1, \cdots, \theta_n\xi_n)=0$, $\xi \in \boldsymbol{R}^n$, $\theta_j^{2p}=1$, $j=1, \cdots, n$).

Example 2. Any weakly hyperbolic operator with respect to H is a weak 1-evolution operator (of S-type) and *vice versa*. Schrödinger operator $D_t - \Delta$ is a weak 2-evolution operator of S-type.

Theorem 3. *The following conditions on the polynomial P and a rational number $\alpha>0$ are equivalent.*

(a) *There exists a non-trivial solution $\psi(t, x) \in C^{l-1}(R_t; \mathcal{S}'_x)$ of $P(D_t, D_x)\psi=0$ such that $D_t^j\hat{\psi}(t, \zeta)$, $j=0, \cdots, l-1$, the Fourier transforms of $D_t^j\psi(t, x)$ with respect to x, can be extended to entire holomorphic functions of $\zeta \in \boldsymbol{C}^n$. Moreover, for some constants $c, M>0$ and some $\beta>0$, α/β being an integer, we have*

$$(3) \qquad |D_t^j\hat{\psi}(t, \zeta)| \leqq M \exp\{c|\zeta|^{2\beta} + |t|\,\Phi_\alpha(\zeta)\}, \ t \in \boldsymbol{R}, \ j=0, \cdots, l-1,$$

where $\Phi_\alpha(\zeta)$ is a non-negative, continuous function of ζ and

$$(4) \qquad \Phi_\alpha(\zeta) - \phi_\alpha(\zeta) = o(|\zeta|^\alpha)$$

with $\phi_\alpha(\zeta)$ that is continuous and positively homogeneous of degree α satisfying

$$(5) \qquad \phi_\alpha(\theta\xi)=0, \ {}^\forall\xi \in \boldsymbol{R}^n, \ \theta \in \boldsymbol{C}\backslash 0, \ \arg\theta = \frac{k\pi}{\beta}, \ k=0, 1, \cdots,$$

$$(\text{resp. } \phi_\alpha(\theta_1\xi_1, \cdots, \theta_n\xi_n)=0, \ {}^\forall\xi \in \boldsymbol{R}^n, \ \theta_j^{2\alpha}=1).$$

(b) *One of the non-constant irreducible factors of P defines a weak α-evolution operator (resp. of S-type).*

A similar result holds when one considers solutions in the half space H.

Theorem 4. *The following conditions on P and a rational number $\alpha>0$ are equivalent.*

(a) *There exists a non-trivial solution $\psi(t, x) \in C^{l-1}([0, \infty); \mathcal{S}'_x)$ of $P(D_t, D_x)\psi=0$ such that $D_t^j\hat{\psi}(t, \zeta)$, $j=0, \cdots, l-1$, can be extended to entire holomorphic functions of $\zeta \in \boldsymbol{C}^n$. And for some $c, M>0$ and some $\beta>0$, α/β being an even integer, ψ satisfies* (3) *for any $t \geqq 0$ with some Φ_α and ϕ_α satisfying* (4) *and* (5). (*Here, we do not consider the condition in the parenthesis of* (5).)

(b) *One of the non-constant irreducible factors of P defines a weak α-evolution operator with respect to H.*

As for a slab domain $\Omega = \{(t, x); a<t<b, x \in \boldsymbol{R}^n\}$, we have

Theorem 5. *The followings are equivalent for P and a rational number $\alpha>0$.*

(a) *There exists a non-trivial solution $\psi(t, x) \in C^{l-1}((a, b); \mathcal{S}'_x)$ of $P(D_t, D_x)\psi=0$ such that $D_t^j\hat{\psi}(t, \zeta)$, $j=0, \cdots, l-1$, can be extended to entire holomorphic functions of $\zeta \in \boldsymbol{C}^n$. Moreover for some $c, M>0$ and some points $a<t_1<t_2<b$, ψ satisfies*

$$(6) \qquad |D_t^j\hat{\psi}(t, \zeta)| \leqq Me^{c\Phi_\alpha(\zeta)}, \quad t=t_1, t_2, \quad j=0, \cdots, l-1,$$

where Φ_α with some ϕ_α assumes the same conditions as in (a), *Theorem* 3, *with $\alpha=\beta$.*

(b) *The same statement as* (b) *in Theorem* 3 *is valid.*

Our method of proving these theorems is very similar to that used by F. John, except that we need to refine some of his arguments.

2. Propagation of support and spreading of wave packet. We now seek some meanings in x-space implied by (3) and (6). First, let us consider the case $\alpha=1$ and the implication (a)$\Rightarrow$(b) in Theorem 5. Then, by putting $\Phi_1(\zeta)=|\mathrm{Im}\,\zeta|+O(|\zeta|^{1-\varepsilon})$, $\varepsilon>0$ and $\phi_1(\zeta)=|\mathrm{Im}\,\zeta|$, an application of the classical Paley-Wiener Theorem gives the result due to F. John [4]. (More precise informations on the location of support were obtained by S. Matsuura [5].)

When $\alpha\geqq 2$, we shall consider the solutions that decay exponentially as $|x|\to\infty$. Hence it might be granted to call them *wave packets*, apart from their original physical meaning.

Lemma 6. *Let $u\in C^\infty(R)$, $p>1$, $0<\varepsilon<1$ and $a>0$. Define ε' and p' by $\varepsilon'=\frac{p'}{p}\varepsilon$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then for some $C>0$, u satisfies*

$$|u^{(j)}(x)|\leqq M_j \exp\left\{-\frac{1}{p}|x/a|^p+C|x|^{p-\varepsilon}\right\},\qquad j=0,1,\cdots,\tag{7}$$

if and only if $\hat{u}(\zeta)$ can be extended to an entire holomorphic function of $\zeta\in\boldsymbol{C}$ and with some $C'>0$, satisfies

$$|\hat{u}(\zeta)|\leqq M'_N(1+|\zeta|)^{-N}\exp\left\{\frac{1}{p'}|a\,\mathrm{Im}\,\zeta|^{p'}+C'|\mathrm{Im}\,\zeta|^{p'-\varepsilon'}\right\},\tag{8}$$
$$N=0,1,\cdots.$$

This lemma is essentially contained in the results of Gel'fand-Shilov and L. Hörmander ([2], [3]). Making a further study of the relation between the constants a, C, M_j and a, C', M'_N, which their general theorems skip over, we obtain some consequences of our theorems.

Corollary 7. *The following condition is necessary and sufficient for one of the non-constant irreducible factors of P to define the weakly hyperbolic operator with respect to H: There exists a non-trivial solution $\psi(t,x)\in C^\infty(\boldsymbol{R}^{n+1})$ of the equation $P\psi=0$ such that for some $a, A_j>0$, $j=0,\cdots,n$ and $q<2$ we have*

$$|D_x^\alpha D_t^k(t,x)|\leqq M_\alpha \exp\left\{\sum_j h(x_j,a,A_j|t|)+A_0|t|^q\right\},$$
$$\text{for any } \alpha=(\alpha_1,\cdots,\alpha_n),\ k=0,\cdots,l-1,$$

where

$$h(x_j,a,b)=\begin{cases}-\dfrac{1}{2a}(|x_j|-b)^2, & |x_j|\geqq b,\\ 0, & |x_j|\leqq b.\end{cases}$$

Corollary 8. *One of the non-constant irreducible factors of P defines a weak 2-evolution operator of S-type if and only if the equation*

$P\psi=0$ in Ω admits a non-trivial solution $\psi(t,x)\in C^\infty(\Omega)\cap C^{l-1}((a,b);\mathcal{S}'_x)$ such that with some positive constants $A_j, C, 0<\varepsilon\leqq 1, N$ and $a<t_1<t_2<b$, it satisfies

$$|D_x^\alpha D_t^k e^{-\Sigma x_j^2/2a_j} \underset{x}{*} \psi(t,x)|\leqq M_\alpha \prod_j s(a_j, A_j)^N$$
$$\times \exp\left\{-\frac{1}{2}\sum |x_j/s(a_j,A_j)|^2+\sum Cs(a_j,A_j)^{-4+2\varepsilon}|x_j|^{2-\varepsilon}\right\},$$
$$\forall a_j>0,\ \forall\alpha,\ t=t_1,t_2,\ k=0,\cdots,l-1,$$

where

$$s(a_j,A_j)=\sqrt{a_j+4A_j^2/a_j}.$$

Finally we note that, putting $\alpha=2$, $\beta=1$ and $\phi_2(\zeta)=|\mathrm{Im}\,\zeta|^2$ in (a), Theorem 4, we have an important type of solutions that actually appears to the heat equation $\dfrac{\partial}{\partial t}-\Delta$.

References

[1] Ehrenpreis, L.: Fourier Analysis in Several Complex Variables. Wiley-Int. Publ., New York (1970).

[2] Gel'fand, I. M., and G. E. Shilov: Generalized Functions. vol. 3, Academic Press (1967).

[3] Hörmander, L.: La transformation de Legendre et la théorème de Paley-Wiener. C. R. Acad. Sci. Paris, **240**, 392–395 (1955).

[4] John, F.: Non-admissible data for differential equations with constant coefficients. Comm. Pure Appl. Math., **10**, 391–398 (1957).

[5] Matsuura, S.: On the propagation of support of solutions to general systems of partial differential equations. Lecture Notes in Physics, **39**, 380–386 (1975).

[6] Mizohata, S.: Some remarks on the Cauchy problems. J. Math. Kyoto Univ., **1**, 109–127 (1961).