32. *Micro-Local Cauchy Problems and Local Boundary Value Problems*

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In these notes we present existence theorems of micro-local Cauchy problems for pseudo-differential operators of Fuchsian type and of local boundary value problems for a class of linear partial differential operators. These theorems are proved by applying the following; first, the Cauchy-Kovalevskaja theorem in the sense of Bony-Schapira [2] for pseudo-differential operators (of Fuchsian type), which we mention in § 1; and secondly, the method of analytic continuation developed in Kashiwara-Kawai [4].

§ 1. The Cauchy-Kovalevskaja theorem for pseudo-differential operators of Fuchsian type. Let $(t,z)=(t,z_1,\cdots,z_n)\in X=\boldsymbol{C}\times\boldsymbol{C}^n$. We use the notation $D_t=\partial/\partial t$ and $D_z^\alpha=(\partial/\partial z_1)^{\alpha_1}\cdots(\partial/\partial z_n)^{\alpha_n}$ for $\alpha=(\alpha_1,\cdots,\alpha_n)$. Let

$$P=t^kD_t^m+A_1(t,z,D_z)t^{k-1}D_t^{m-1}+\cdots+A_k(t,z,D_z)D_t^{m-k}$$
$$+\cdots+A_m(t,z,D_z)$$

be a pseudo-differential operator of finite order in the sense of Sato-Kawai-Kashiwara [6] which is defined on an open subset of the cotangential projective bundle P^*X of X. We assume the following conditions:

(A.1) $0\leqq k\leqq m$;

(A.2) ord $A_j(t,z,D_z)\leqq j$ for $j=1,\cdots,m$;

(A.3) ord $A_j(0,z,D_z)\leqq 0$ for $j=1,\cdots,k$.

Then P is said to be of *Fuchsian type with weight* $m-k$ (cf. Baouendi-Goulaouic [1] and Tahara [7]). We set

$$a_j(z,\zeta)=\sigma_0(A_j(0))(z,\zeta) \qquad \text{for } j=1,\cdots,k,$$

where σ_0 denotes the principal symbol of order 0, and $(z,\zeta\infty)$ is a point of $P^*\boldsymbol{C}^n$. The *indicial equation* associated with P is defined by

$$\lambda(\lambda-1)\cdots(\lambda-m+1)+\lambda(\lambda-1)\cdots(\lambda-m+2)a_1(z,\zeta)$$
$$+\cdots+\lambda(\lambda-1)\cdots(\lambda-m+k+1)a_k(z,\zeta)=0,$$

and its roots are called the *characteristic exponents* of P, which we denote by

$$\lambda=0,\cdots,m-k-1,\ \lambda_1(z,\zeta),\cdots,\lambda_k(z,\zeta).$$

For the sake of simplicity, we assume that $A_j(t,z,D_z)$ is defined on a neighborhood of $\mathring{\omega}$ for $j=1,\cdots,m$, where

$$\omega=\{(t, z, \zeta\infty)\in \boldsymbol{C}\times P^*\boldsymbol{C}^n\,;|t|<T,\ z\in U,\\ |\zeta_j|<c_0|\zeta_1| \text{ for } j=2,\cdots,n\}$$

with $T>0$, $c_0>0$, and a relatively compact open subset U of $\boldsymbol{C}^n$.

Let $h\in\boldsymbol{C}$, and set $H=\{z\in\boldsymbol{C}^n\,;z_1=h\}$. Let Ω be an open convex subset of U, and assume that Ω is "c_0-H-plat" in the sense of Bony-Schapira [2]; that is, if $z\in\Omega$, $w\in H$ and $c_0|z_j-w_j|\leqq|z_1-w_1|$ for $j=2,\cdots,n$, then $w\in\Omega\cap H$. Let $f(t,z)$ be a holomorphic function defined on $W=\{(t,z)\in\boldsymbol{C}\times\Omega\,;|t|<T\}$. If q is a positive integer, there is a unique holomorphic function $g(t,z)$ on W such that;

$$\begin{cases} D_{z_1}^q g(t,z)=f(t,z), \\ D_{z_1}^j g|_{z_1=h}=0 & \text{for } j=0,\cdots,q-1. \end{cases}$$

Then we denote $g(t,z)$ by $(D_{z_1}^{-q})_H f(t,z)$. Let

$$A_j(t,z,D_z)=\sum_{\substack{\alpha_1\in\boldsymbol{Z}\\ \alpha_2,\cdots,\alpha_n\geqq 0}} a_{j,\alpha}(t,z)D_z^\alpha,$$

and let

$$(A_j)_H f(t,z)=\sum_{\alpha_1,\cdots,\alpha_n\geqq 0} a_{j,\alpha}(t,z)D_z^\alpha f(t,z) \\ +\sum_{\substack{\alpha_1<0\\ \alpha_2,\cdots,\alpha_n\geqq 0}} a_{j,\alpha}(t,z)(D_{z_1}^{\alpha_1})_H D_{z_2}^{\alpha_2}\cdots D_{z_n}^{\alpha_n} f(t,z).$$

By applying the argument of [2] regarding t as a holomorphic parameter, we find that $(A_j)_H f(t,z)$ is holomorphic on W. We set

$$P_H f(t,z)=t^k D_t^m f(t,z)+(A_1)_H t^{k-1}D_t^{m-1}f(t,z)+\cdots+(A_m)_H f(t,z).$$

Then $P_H f(t,z)$ is also holomorphic on W. Let fix a point $z_0\in\Omega\cap H$, and set

$$\Omega_s=\{s(z-z_0)+z_0\,;z\in\Omega\}\qquad\text{for } 0<s\leqq 1.$$

Now we assume the following:

(A.4) $\lambda_j(z,\zeta)\notin\{i\in\boldsymbol{Z}\,;i\geqq m-k\}$ for $j=1,\cdots,k$, and $(0,z,\zeta\infty)\in\bar{\omega}$.

Under the above assumptions, we have the following

Theorem 1. *If the diameter of Ω is sufficiently small, there exists a positive number δ such that for any holomorphic function $f(t,z)$ on $\{(t,z)\in\boldsymbol{C}\times\Omega_s\,;|t|<T'\}$ with $0<T'\leqq T$ and $0<s\leqq 1$, and for any holomorphic functions $v_0(z),\cdots,v_{m-k-1}(z)$ on Ω_s, there exists a unique holomorphic solution $u(t,z)$ of the Cauchy problem*

$$\begin{cases} P_H u(t,z)=f(t,z), \\ D_t^j u|_{t=0}=v_j(z) & \text{for } j=0,\cdots,m-k-1, \end{cases}$$

and $u(t,z)$ is holomorphic on

$$\bigcup_{0<s'<s}(\{t\in\boldsymbol{C}\,;|t|<\min(\delta(s-s')^p,T')\}\times\Omega_{s'}),$$

where $p=\min(k+1,m)$.

Remark 1. When P is a partial differential operator, this theorem has been proved in [1] and [7]. In [7], Fuchsian systems of partial differential equations are also treated. Our proof of Theorem 1 depends on the techniques in [1] and [2].

§2. Micro-local Cauchy problems. Let $M=\boldsymbol{R}\times\boldsymbol{R}^n\ni(t,x)=(t,$

$x_1, \cdots, x_n)$ and $N=\boldsymbol{R}^n \ni x$. Under the injection $\iota: N \to M$ defined by $\iota(x)=(0, x)$, we regard $N \subset M$. The map

$$\rho: \sqrt{-1}S^*M \underset{M}{\times} N - \sqrt{-1}S_N^*M \to \sqrt{-1}S^*N$$

is defined by

$$\rho(0, x, \sqrt{-1}(\tau dt + \langle \xi, dx \rangle)\infty) = (x, \sqrt{-1}\langle \xi, dx \rangle \infty).$$

Let

$$P = t^k D_t^m + A_1(t, x, D_x) t^{k-1} D_t^{m-1} + \cdots + A_k(t, x, D_x) D_t^{m-k} + \cdots + A_m(t, x, D_x)$$

be a pseudo-differential operator of Fuchsian type with weight $m-k$ defined on a neighborhood of $\rho^{-1}(x_0^*)$, where $x_0^* = (x_0, \sqrt{-1}\langle \xi_0, dx \rangle \infty)$ is a point of $\sqrt{-1}S^*N$. We assume the following conditions:

(A.4)′ $\lambda_j(x_0^*) \notin \{i \in \boldsymbol{Z}; i \geqq m-k\}$ for $j=1, \cdots, k$;

(B.1) $\sigma_m(P)(t, x, \tau, \xi) = t^k p_m(t, x, \tau, \xi)$ for some analytic function p_m;

(B.2) Let $\tau = \tau_1(t, x, \xi), \cdots, \tau_m(t, x, \xi)$ be the roots of $p_m(t, x, \tau, \xi) = 0$. Then for some $\varepsilon > 0$, $t \times \operatorname{Im}(\tau_j(t, x, \xi)) \geqq 0$ for $j=1, \cdots, m$, $-\varepsilon < t < \varepsilon$, $|x - x_0| < \varepsilon$, $\xi \in \boldsymbol{R}^n - \{0\}$, and $|\xi - \xi_0| < \varepsilon$.

We denote by $\mathcal{C}_M$ and $\mathcal{C}_N$ the sheaves of microfunctions associated with M and N respectively. We abbreviate

$$\rho_!\left(\mathcal{C}_M | \sqrt{-1}S^*M \underset{M}{\times} N - \sqrt{-1}S_N^*M\right)$$

to $\rho_!\mathcal{C}_M$. $(\rho_!\mathcal{C}_M)_{x_0^*}$ denotes the set of microfunctions defined on a neighbourhood of $\rho^{-1}(x_0^*)$ "having t as a real analytic parameter" (cf. [6]).

Theorem 2. *Under the above conditions, there exists, for any* $f(t, x) \in (\rho_!\mathcal{C}_M)_{x_0^*}$ *and for any* $v_0(x), \cdots, v_{m-k-1}(x) \in (\mathcal{C}_N)_{x_0^*}$, *a solution* $u(t, x) \in (\rho_!\mathcal{C}_M)_{x_0^*}$ *of the Cauchy problem*

$$\begin{cases} Pu(t, x) = f(t, x) & \text{on } \rho^{-1}(x_0^*), \\ D_t^j u|_{t=0} = v_j(x) & \text{at } x_0^* \text{ for } j=0, \cdots, m-k-1. \end{cases}$$

Remark 2. It has been proved in [7] that the Cauchy problem in the framework of the hyperfunctions is well-posed for hyperbolic partial differential operators of Fuchsian type satisfying condition (A.4)′. In [7], Cauchy problems in the framework of the microfunctions for Fuchsian micro-hyperbolic systems of pseudo-differential equations are also dealt with in the homogeneous case (i.e. $Pu=0$).

Example 1. Let $x_0^* = (0, \sqrt{-1}dx_1\infty) \in \sqrt{-1}S^*N$, and let

$$P = t(D_t - \sqrt{-1}tD_{x_1}) - Q(t, x, D_x),$$

where Q is a pseudo-differential operator defined on a neighborhood of $\rho^{-1}(x_0^*)$, of order at most 0. We assume:

$$\sigma_0(Q)(x_0^*) \neq 0, 1, 2, \cdots.$$

Then the homomorphism

$$P: (\rho_!\mathcal{C}_M)_{x_0^*} \to (\rho_!\mathcal{C}_M)_{x_0^*}$$

is surjective.

§3. Local boundary value problems. Let

$$P=D_t^m+A_1(t, x, D_x)D_t^{m-1}+\cdots+A_m(t, x, D_x)$$

be a linear partial differential operator of order m with real analytic coefficients defined on a neighborhood of $(t, x)=(0, 0)$. We denote the roots of $\sigma_m(P)(t, x, \tau, \xi)=0$ by $\tau=\tau_1(t, x, \xi), \cdots, \tau_m(t, x, \xi)$. Let M and N be as above. We put $\sqrt{-1}S^*N=N\times\sqrt{-1}S^{n-1}$ where S^{n-1} is the $(n-1)$-sphere.

Let I be an open subset of S^{n-1}, and assume the following conditions for some integer m' with $1\leqq m'\leqq m$:

(C.1) For any compact subset K of I, there is a positive number ε_K such that;

$\operatorname{Im}\tau_j(t, x, \xi)\geqq 0$ for $j=1, \cdots, m'$, $0\leqq t<\varepsilon_K$, $|x|<\varepsilon_K$, and $\xi\infty\in K$;

(C.2) $\{\tau_j(0, 0, \xi)\,;\ j=1, \cdots, m'\}$ and $\{\tau_j(0, 0, \xi)\,;\ j=m'+1, \cdots, m\}$ are disjoint from each other if $\xi\infty\in I$.

Theorem 3. *Suppose that P satisfies the above conditions. Then, if $v_j(x)$ is a hyperfunction on $\{|x|<a\}$ with $a>0$, and if the singular spectrum of $v_j(x)$ is contained in $\{|x|<a\}\times\sqrt{-1}I$ for $j=0, \cdots, m'-1$, there exists a hyperfunction $u(t, x)$ on $\{(t, x)\in M\,;\ 0<t<a', |x|<a'\}$ for some a' with $0<a'\leqq a$ such that*

$$\begin{cases} Pu(t, x)=0, \\ D_t^j u|_{t\to+0}=v_j(x) \qquad \text{for } j=0, \cdots, m'-1, \end{cases}$$

where $|_{t\to+0}$ denotes the boundary value from $t>0$ in the sense of Komatsu-Kawai [5].

Remark 3. When $m'=m$, this theorem has been proved in Kaneko [3].

Example 2. Let

$$P=(D_t^2+\Delta_x)(D_t^2-t\Delta_x)+Q(t, x, D_t, D_x),$$

where $\Delta_x=D_{x_1}^2+\cdots+D_{x_n}^2$, and Q is a linear partial differential operator with real analytic coefficients defined on a neighborhood of $(t, x)=(0, 0)$, of order at most 3. Then P satisfies (C.1) and (C.2) with $I=S^{n-1}$ and $m'=3$.

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