36. *Studies on Holonomic Quantum Fields. XIV*

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The present article is a direct continuation of our preceding note [1], where deformation theory was discussed in connection with the Riemann-Hilbert problem for Euclidean Dirac equations. We are particularly interested in the step function limit of the matrix $M(\xi)$; in other words the Green's function $w(x,x')$ is now required to be multi-valued, having a monodromic property $w(x,x')\mapsto e^{2\pi i L_\nu}w(x,x')$ when continued around 2-codimensional submanifolds ("Bags") $B_\nu=\{f_\nu=0,\bar{f}_\nu=0\}$. Formally the variational formula XIII-(7) [1] then takes the form

$$(1)\qquad \frac{1}{2\pi i}\delta w(x,x')=\sum_\nu\int_{X^{\mathrm{Euc}}}d^s y\cdot w(x,y)\Delta_\nu(y)L_\nu w(y,x')$$

$$\Delta_\nu(y)=\frac{1}{2i}(\partial\!\!\!/ f_\nu(y)\delta\bar{f}_\nu(y)-\delta f_\nu(y)\partial\!\!\!/\bar{f}_\nu(y))\delta(f_{\nu 1}(y))\delta(f_{\nu 2}(y))$$

with $f_\nu(y)=f_{\nu1}(y)+if_{\nu2}(y)$. However the meaning of (1) needs to be made precise, since $w(x,x')$ has a regular singularity along B_ν. In this note we perform this procedure in the 2-dimensional (massless and massive) case, and show that the resulting equations are exactly those obtained previously ((2.3.38) in [2] and (3.3.53) in [3]).

We use the following convention:

$$\gamma^1=\begin{pmatrix} & 1\\ 1 & \end{pmatrix},\ \gamma^2=\begin{pmatrix} & -i\\ i & \end{pmatrix},\ \partial\!\!\!/=\begin{pmatrix} & \partial\\ \bar\partial & \end{pmatrix},\ \partial=\partial_1-i\partial_2,\ \bar\partial=\partial_1+i\partial_2.$$

1. The Riemann-Hilbert problem for the Euclidean Dirac equation in the sense of [1] has a special feature when the space dimension is 2 and the mass vanishes. Let us restate the problem in this case. As in [1] we denote by D^+ a bounded domain in $X^{\mathrm{Euc}}=\boldsymbol{R}^2$, and let $D^-=X^{\mathrm{Euc}}-\bar{D}^+$, $\partial D^+=\Gamma$. We set $z=(x^1+ix^2)/2$, $\bar z=(x^1-ix^2)/2$. Given a real analytic $N\times N$ matrix M on Γ, we are to find a $2N\times 2N$ matrix

$$w=\begin{pmatrix} w_1 & w_2\\ w_3 & w_4\end{pmatrix}$$

such that

$$(2)\quad \text{(i)}\quad -\begin{pmatrix} & \partial\\ \bar\partial & \end{pmatrix}w(z,\bar z;z',\bar z')=\delta(x^1-x'^1)\delta(x^2-x'^2)\quad (x,x'\notin\Gamma)$$

$$\text{(ii)}\quad |w(z,\bar z;z',\bar z')|=O\left(\frac{1}{|z|}\right)\quad (|z|\to\infty)$$

$$\text{(iii)}\quad w(\zeta^+,\bar\zeta^+;z',\bar z')=M(\zeta,\bar\zeta)w(\zeta^-,\bar\zeta^-;z',\bar z')\quad (\zeta,\bar\zeta)\in\Gamma$$

where

$$w(\zeta^{\pm},\bar{\zeta}^{\pm};z',\bar{z}')=\lim_{D^{\pm}\ni(z,\bar{z})\to(\zeta,\bar{\zeta})} w(z,\bar{z};z',\bar{z}').$$

Now from (2)-(i) and (ii) we see immediately that $w_1\equiv 0$, $w_4\equiv 0$, and that

$$(3)\qquad w_2=-\frac{1}{4\pi}\frac{1}{z-z'}Y(z,z';\Gamma,M)$$

$$w_3=-\frac{1}{4\pi}\frac{1}{\bar{z}-\bar{z}'}\overline{Y(z,z';\Gamma,\bar{M})}.$$

Here $Y(z,z';\Gamma,M)=Y(z,z')$ is a holomorphic matrix defined on $(\boldsymbol{P}_C^1-\Gamma)\times(\boldsymbol{P}_C^1-\Gamma)$ characterized by either of the following (with the abbreviation $M(\zeta)=M(\zeta,\bar{\zeta})$):

$$(4)\qquad Y(z,z)=1,\ Y(\zeta^+,z')=M(\zeta)Y(\zeta^-,z')\quad(\zeta\in\Gamma)$$

$$(4)'\qquad Y(z,z)=1,\ Y(z,\zeta'^+)=Y(z,\zeta'^-)M(\zeta')^{-1}\quad(\zeta'\in\Gamma).$$

This is the ordinary Riemann-Hilbert problem corresponding to a continuous "monodromy matrix" M. If M is sufficiently close to 1 the solution $Y(z,z')$ exists uniquely, which is shown to be an invertible matrix for any (z,z'). As a consequence of this and the characteristic property (4) we obtain the simple relation

$$(5)\qquad Y(z,z')Y(z',z'')=Y(z,z''),\qquad Y(z',z)=Y(z,z')^{-1}.$$

Thanks to the "splitting property" (5) the variational formulas for $Y(z,z')$ simplify a great deal. By applying XIII-(7) [1] the $M(\zeta)$-preserving variation of $Y(z,z')$ along a vector field $\delta\rho(\zeta)\cdot\partial_\zeta+\delta\bar{\rho}(\zeta)\cdot\bar{\partial}_\zeta$ is given by

$$(6)\qquad \delta Y(z,z')$$

$$=\frac{1}{2\pi i}\int_\Gamma d\zeta\cdot\delta\rho(\zeta)\left(\frac{1}{z-\zeta}-\frac{1}{z'-\zeta}\right)Y(z,\zeta^+)\cdot\partial_t M(\zeta)\cdot Y(\zeta^-,z')$$

with

$$\partial_t=\partial_\zeta+4\left(\frac{d\bar{\zeta}}{ds}\right)^2\bar{\partial}_\zeta=\left(\frac{d\zeta}{ds}\right)^{-1}\frac{d}{ds}$$

denoting the tangential component of ∂_ζ (s is the arc length such that $4|d\zeta/ds|^2=1$). Setting

$$(7)\qquad A(z;\zeta)=Y(z,\zeta^+)\cdot\partial_t M(\zeta)\cdot Y(\zeta^-,z)$$

and applying (5) we may write (6) as

$$(8)\qquad \delta Y(z,z')=\left(\frac{1}{2\pi i}\int_\Gamma d\zeta\cdot\delta\rho(\zeta)\left(\frac{1}{z-\zeta}-\frac{1}{z'-\zeta}\right)A(z;\zeta)\right)\cdot Y(z,z')$$

$$=Y(z,z')\cdot\left(\frac{1}{2\pi i}\int_\Gamma d\zeta\cdot\delta\rho(\zeta)\left(\frac{1}{z-\zeta}-\frac{1}{z'-\zeta}\right)A(z';\zeta)\right).$$

Likewise we rewrite the equations of Euclidean covariance XIII-(23), (24) [1] and single out expressions for $\partial_z Y,\partial_{z'}Y$ from them. The results read

$$(9)\qquad \partial_z Y(z,z')=\left(-\frac{1}{2\pi i}\int_\Gamma d\zeta\frac{A(z;\zeta)}{z-\zeta}\right)\cdot Y(z,z')$$

$$=Y(z,z')\cdot\left(-\frac{1}{2\pi i}\int_\Gamma d\zeta\frac{A(z';\zeta)}{z-\zeta}\right)$$

(10) $$\partial_{z'}Y(z,z')=\left(\frac{1}{2\pi i}\int_\Gamma d\zeta\frac{A(z;\zeta)}{z'-\zeta}\right)\cdot Y(z,z')$$
$$=Y(z,z')\cdot\left(\frac{1}{2\pi i}\int_\Gamma d\zeta\frac{A(z';\zeta)}{z'-\zeta}\right).$$

Equations (8), (9) and (10) constitute an analogue of the total differential equation (2.3.38) of [2] for the solution of the Riemann's problem. From (7)–(10) it is straightforward to calculate the variation of the coefficient matrix $A(z;\zeta)$. We thus obtain the following continuous monodromy version of the Schlesinger's equations (2.3.43) [2].

(11) $$\partial_z A(z;\zeta)=\frac{1}{2\pi i}\int_\Gamma d\zeta'\frac{1}{z-\zeta'}[A(z;\zeta),A(z;\zeta')]$$

(12) $$\delta A(z,\zeta)=-\frac{1}{2\pi i}\int_\Gamma d\zeta'\left(\frac{\delta\rho(\zeta)-\delta\rho(\zeta')}{\zeta-\zeta'}-\frac{-\delta\rho(\zeta')}{z-\zeta'}\right)[A(z;\zeta),A(z;\zeta')]$$
$$-2(\partial_t\delta\rho)(\zeta)\cdot A(z;\zeta).$$

The original equations (2.3.38), (2.3.43) [2] are reproduced by passing to the limiting case where

(13) $$\frac{dM}{ds}M^{-1}=-2\pi i\sum_{\nu=1}^{n}L_\nu\delta(s-s_\nu).$$

In this case of the original Riemann's problem, the solution has a regular singularity at $z'=a_\nu=\zeta(s_\nu)$: $Y(z,z')=\Phi_\nu(z,z')\cdot(z'-a_\nu)^{-L_\nu}$, $\Phi_\nu(z,z')$ being holomorphic and invertible at $z'=a_\nu$. We have then

(14) $$A(z;\zeta)=-2\pi i\left(\frac{d\zeta}{ds}\right)^{-1}\sum_{\nu=1}^{n}[Y(z,z')L_\nu Y(z',z)]\,|_{z'=a_\nu}\cdot\delta(s-s_\nu)$$
$$=2\pi i\sum_{\nu=1}^{n}A_\nu(z)\cdot\left(\frac{d\zeta}{ds}\right)^{-1}\delta(s-s_\nu)$$

where

$$A_\nu(z)=\Phi_\nu(z,z')\cdot(z'-a_\nu)^{-L_\nu}(-L_\nu)(z'-a_\nu)^{L_\nu}\Phi_\nu(z,z')^{-1}\,|_{z'=a_\nu}$$
$$=\Phi_\nu(z,a_\nu)(-L_\nu)\Phi_\nu(z,a_\nu)^{-1}.$$

If we write $\delta a_\nu=\delta\rho(a_\nu)$, (8)–(10) and (11), (12) reduce respectively to

(15) $$\delta Y(z,z')=\left(\sum_{\nu=1}^{n}\left(\frac{1}{z-a_\nu}-\frac{1}{z'-a_\nu}\right)A_\nu(z)\,\delta a_\nu\right)\cdot Y(z,z')$$

(16) $$\partial_z Y(z,z')=\left(-\sum_{\nu=1}^{n}\frac{A_\nu(z)}{z-a_\nu}\right)\cdot Y(z,z')$$

(17) $$\partial_{z'}Y(z,z')=\left(\sum_{\nu=1}^{n}\frac{A_\nu(z)}{z'-a_\nu}\right)\cdot Y(z,z')$$

(18) $$\partial_z A_\mu(z)=\sum_{\nu(\neq\mu)}[A_\mu(z),A_\nu(z)]\frac{1}{z-a_\nu}$$

(19) $$\delta A_\mu(z)=-\sum_{\nu(\neq\mu)}[A_\mu(z),A_\nu(z)]\left(\frac{\delta a_\mu-\delta a_\nu}{a_\mu-a_\nu}-\frac{-\delta a_\nu}{z-a_\nu}\right)$$

where we have used

$$2\pi i\sum_{\mu=1}^{n}A_\mu(z)\delta\left(\left(\frac{d\zeta}{ds}\right)^{-1}\delta(s-s_\mu)\right)=-2(\partial_t\delta\rho)(\zeta)\cdot A(z;\zeta).$$

2. Here we shall reformulate the monodromy problem for the massive Euclidean Dirac equation [3], [4] applying the variational method developed in [1]. The notations used in this paragraph sometimes differs from those in [3], [4].

Let $a_1, \cdots, a_n$ be distinct n points in X^{Euc}, and let $L_1, \cdots, L_n$ be $N\times N$ matrices. We set $X'=X^{\mathrm{Euc}}-\{a_1, \cdots, a_n\}$ and denote by $\tilde{X}'$ the universal covering of X'. A $2N\times 2N$ matrix valued function $w(x, x')=w(x, x'; a_1, \cdots, a_n; L_1, \cdots, L_n)$ defined for $(x, x')\in \tilde{X}'\times\tilde{X}'$ is called the Green's function for the Riemann data $(a_1, \cdots, a_n; L_1, \cdots, L_n)$ if it satisfies the following:

(i) $(-\partial\!\!\!/_x+m)w(x, x')=\delta^2(x-x')$.

(ii) $|w(x, x')|=O(e^{-m|x|})$ when x tends to infinity in a finite sector.

(iii) For every $x'\in\tilde{X}'$ there exists $2N\times 2N$ matrices $u_\nu(x, x')=u_\nu(x, x'; a_1, \cdots, a_n; L_1, \cdots, L_n)$ and $u_\nu(x, x')=u^*_\nu(x, x'; a_1, \cdots, a_n; L_1, \cdots, L_n)$ which are defined and real analytic at $x=a_\nu$ so that the local expression of $w(x, x')$ at $x=a_\nu$ reads

(20) $$w(x, x')=z(x-a_\nu)^{L_\nu}u_\nu(x, x')+\bar{z}(x-a_\nu)^{-L_\nu}u^*_\nu(x, x').$$

Here we have set

$$z(x)=m\frac{x^1+ix^2}{2} \quad\text{and}\quad \bar{z}(x)=m\frac{x^1-ix^2}{2}.$$

The precise meaning of (i) is as follows: $(-\partial\!\!\!/_x+m)w(x, x')=\delta^2(\pi(x)-\pi(x'))$ (if x is near x'), $=0$ (otherwise). Here $\pi: \tilde{X}'\to X'$ is the covering map.

For sufficiently small $L_1, \cdots, L_n$ the Green's function for the Riemann data $(a_1, \cdots, a_n; L_1, \cdots, L_n)$ exists and it is unique. It is also characterized by the following alternative.

(i)′ $w(x, x')(\overleftarrow{\partial\!\!\!/}_{x'}+m)=\delta^2(x-x')$.

(ii)′ $|w(x, x')|=O(e^{-m|x'|})$ $(|x'|\to\infty)$.

(iii)′ $w(x, x')=v_\nu(x, x')z(x'-a_\nu)^{-L_\nu}+v^*_\nu(x, x')\bar{z}(x'-a_\nu)^{L_\nu}$ $(|x'-a_\nu|\ll 1)$.

The variational formula for $w(x, x')$ takes the following form.

(21) $$\frac{m}{4\pi}\delta w(x, x')=\sum_\nu v_\nu(x, a_\nu)\cdot L_\nu\otimes\begin{pmatrix} & 0\\ -\delta z(a_\nu) & \end{pmatrix}\cdot u_\nu(a_\nu, x') +\sum_\nu v^*_\nu(x, a_\nu)\cdot L_\nu\otimes\begin{pmatrix} & \delta\bar{z}(a_\nu)\\ 0 & \end{pmatrix}\cdot u^*_\nu(a_\nu, x').$$

Now assume that L_ν is non-singular ($\nu=1, \cdots, n$), and set

(22) $$w_\nu(x)=\frac{4\pi}{m}v_\nu(x, a_\nu)\begin{pmatrix}0\\ \Gamma(L_\nu)^{-1}\end{pmatrix},\quad w^*_\nu(x)=\frac{4\pi}{m}v^*_\nu(x, a_\nu)\begin{pmatrix}\Gamma(-L_\nu)^{-1}\\ 0\end{pmatrix}.$$

(23) $$\overline{w}_\nu(x')=\frac{4\pi}{m}(\Gamma(-L_\nu)^{-1}, 0)u_\nu(a_\nu, x'),$$
$$\overline{w}^*_\nu(x')=-\frac{4\pi}{m}(0, \Gamma(L_\nu)^{-1})u^*_\nu(a_\nu, x'),$$

where $\Gamma(L)$ is the gamma function. Then the solutions $w_\nu(x)$ and $w_\nu^*(x)$ to the Euclidean Dirac equation are characterized by the exponential decreasing property at $|x|\to\infty$ and the following local expansions at $x=a_\mu$.

$$w_\nu(x)=w_{L_\mu-1/2}(x-a_\mu)\delta_{\mu\nu}+w_{L_\mu+1/2}(x-a_\mu)\alpha_{\mu\nu}+\cdots +w^*_{-L_\mu+1/2}(x-a_\mu)\beta_{\mu\nu}+\cdots, \tag{24}$$

$$w_\nu^*(x)= w_{L_\mu+1/2}(x-a_\mu)\beta^*_{\mu\nu}+\cdots +w^*_{-L_\mu-1/2}(x-a_\mu)\delta_{\mu\nu}+w^*_{-L_\mu+1/2}(x-a_\mu)\alpha^*_{\mu\nu}+\cdots. \tag{25}$$

Here we have set

$$w_L(x)=\begin{pmatrix}v_{L-1/2}(x)\\ v_{L+1/2}(x)\end{pmatrix},\qquad w_L^*(x)=\begin{pmatrix}v^*_{L+1/2}(x)\\ v^*_{L-1/2}(x)\end{pmatrix},$$

$v_L(x)=e^{iL\theta}I_L(mr)$ and $v_L^*(x)=e^{-iL\theta}I_L(mr)$ where $z(x)=re^{i\theta}/2$. $\alpha_{\mu\nu}$, $\alpha^*_{\mu\nu}$, $\beta_{\mu\nu}$ and $\beta^*_{\mu\nu}$ are $N\times N$ matrices independent of x. $\overline{w}_\nu(x')$ and $\overline{w}_\nu^*(x')$ are characterized similarly. They have the following local expansions at $x'=a_\mu$.

$$\overline{w}_\nu(x')=\delta_{\nu\mu}\overline{w}_{-L_\mu-1/2}(x'-a_\mu)+\alpha'_{\nu\mu}\overline{w}_{-L_\mu+1/2}(x'-a_\mu)+\cdots +\beta'_{\nu\mu}\overline{w}^*_{L_\mu+1/2}(x'-a_\mu)+\cdots, \tag{26}$$

$$\overline{w}_\nu^*(x')= \beta^{*\prime}_{\nu\mu}\overline{w}_{-L_\mu+1/2}(x'-a_\mu)+\cdots +\delta_{\mu\nu}\overline{w}^*_{L_\mu-1/2}(x'-a_\mu)+\alpha^{*\prime}_{\nu\mu}\overline{w}^*_{L_\mu+1/2}(x'-a_\mu)+\cdots, \tag{27}$$

where

$$\overline{w}_L(x)=(v_{L+1/2}(x),\ -v_{L-1/2}(x))\quad\text{and}\quad \overline{w}_L^*(x)=(v^*_{L-1/2}(x),\ -v^*_{L+1/2}(x)).$$

The variational equation now reads

$$\frac{4\pi}{m}\delta w(x,x')=\sum_\nu w_\nu(x)\frac{\pi}{\sin\pi L_\nu}\overline{w}_\nu(x')\delta z(a_\nu) +\sum_\nu w_\nu^*(x)\frac{\pi}{\sin\pi L_\nu}\overline{w}_\nu^*(x')\delta\bar{z}(a_\nu). \tag{28}$$

If $N=1$ (28) is equivalent to (3.3.53) in [3]. In [3] we have derived (3.3.53) starting from the holonomic system (3.3.20) and the deformation equation (3.3.24). Conversely, the Euclidean covariance of $w(x,x')$ and the variational equation (28) implies the holonomic system for $\tilde{w}(x)=(w_1(x),\cdots,w_n(x))$ given below.

First we prepare several notations. We denote by α,β, etc. the $nN\times nN$ matrices $(\alpha_{\mu\nu})_{\mu,\nu=1,\cdots,n}$, $(\beta_{\mu\nu})_{\mu,\nu=1,\cdots,n}$, etc. We set also $\mathcal{L}=(\delta_{\nu\mu}L_\nu)_{\mu,\nu=1,\cdots,n}$, $z(A)=(\delta_{\mu\nu}z(a_\nu))_{\mu,\nu=1,\cdots,n}$ and $\bar{z}(A)=(\delta_{\mu\nu}\bar{z}(a_\nu))_{\mu,\nu=1,\cdots,n}$. Notice that $\beta\beta^*=1$ and $\partial\tilde{w}(x)/\partial\bar{z}(x)=\tilde{w}^*(x)\beta$, $\tilde{w}^*(x)=(w_1^*(x),\cdots,w_n^*(x))$. Finally we set

$$M_F(a,L)w(x)=z(x-a)\frac{\partial w(x)}{\partial z(x)}-\bar{z}(x-a)\frac{\partial w(x)}{\partial\bar{z}(x)} +\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}w(x)-w(x)\Big(L-\frac{1}{2}\Big).$$

Then we have

$$(-\partial\!\!\!/+m)\tilde{w}(x)=0,\qquad (M_F(a_1,L_1)w_1(x),\cdots,M_F(a_n,L_n)w_n(x)) \tag{29}$$

$$= \tilde{w}(x)[z(A), \alpha] - \tilde{w}^*(x)[\bar{z}(A), \beta],$$

$$\delta\tilde{w}(x) = -\frac{\partial\tilde{w}(x)}{\partial z(x)}\delta z(A) - \frac{\partial\tilde{w}(x)}{\partial\bar{z}(x)}\delta\bar{z}(A) - \tilde{w}(x)[\delta z(A), \alpha] - \tilde{w}^*(x)[\delta\bar{z}(A), \beta].$$

Now the local expansion (24), (25) and the linear equations (29) imply the following non linear equations for α and β.

(30) $$\delta\alpha = [\delta z(A), \alpha^{(1)}] - \alpha[\delta z(A), \alpha] - \beta^*[\delta\bar{z}(A), \beta],$$
$$\delta\beta = -\beta[\delta z(A), \alpha] + [\delta\bar{z}(A), \alpha^*]\beta,$$

(31) $$\alpha + [\mathcal{L}, \alpha] = -[z(A), \alpha^{(1)}] + \alpha[z(A), \alpha] - \beta^*[\bar{z}(A), \beta],$$
$$[\mathcal{L}, \beta] = \beta[z(A), \alpha] + [\bar{z}(A), \alpha^*]\beta.$$

Here $\beta^* = \beta^{-1}$, and $\alpha^{(1)}$ and α^* are to be eliminated by using the algebraic relations (31). If $N=1$ (29)–(31) are equivalent to (3.3.20) and (3.3.24) in [3].

References

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