61. On Some Periodic 4-Transitive Permutation Groups

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1. **Introduction.** In [2], O. H. Kegel determined the locally finite Zassenhaus groups with some additional conditions. By making use of some ideas in the proofs of M. Hall [1] and V. P. Shunkov [4], we shall prove the following theorem allied to Kegel's result.

Theorem. *Let G be a periodic 4-transitive permutation group on a set Ω ($|\Omega|\leqq\infty$). If $G_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}=1$ for any distinct five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of Ω, then G is a finite group and is isomorphic to one of the following groups: $S_4, S_5, S_6, A_6, A_7, M_{11}$ or M_{12}.*

Definitions. Let G be a group. G is called a periodic group if every element of G has finite order. G is called a locally finite group if every finite subset of G generates a finite group. G is called a Frobenius group if G contains a proper subgroup H such that $g^{-1}Hg\cap H=1$ for all $g\in G-H$. Such a subgroup H of the Frobenius group G is called a Frobenius complement of G.

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2. **Proof of Theorem.** In the first place, we prove the following

Lemma. *Let G be a periodic Frobenius group and H a Frobenius complement of G. Then H contains at most one involution.*

Proof. Suppose, by way of contradiction, that H contains two involutions i and j. Let g be an involution in $G-H$. First we show that there exists an involution y in $G-H$ such that $y^{-1}iy=g$. If $|ig|$ (=the order of ig) is even, then we have $ia=ai$ and $ga=ag$ for the involution a in $\langle ig\rangle$. Therefore we have $a\in C_G(i)\subseteq H$, and so we have $g\in C_G(a)\subseteq H$, a contradiction. Hence there exists an element x in $\langle ig\rangle$ such that $x^{-1}ix=g$, because $|ig|$ is odd. Set $ix=y$. Then y is an involution in $G-H$ such that $y^{-1}iy=g$. Similarly, there exists an involution z in $G-H$ such that $z^{-1}jz=g$. Since yz normalizes H and $y^{-1}Hy$ ($=z^{-1}Hz$), we have $yz=1$. Hence we have $i=j$, a contradiction.

Proof of Theorem. Let G be a permutation group satisfying the assumption of Theorem. If G is a finite group, then we know that G is isomorphic to $S_4, S_5, S_6, A_6, A_7, M_{11}$ or M_{12} (cf. [1], [3]). From now on, we shall assume that G is an infinite periodic group and $|\Omega|=\infty$, and prove eventually that this leads to a contradiction. We may assume that $\{1, 2, 3, \cdots\}\subseteq\Omega$.

First suppose that the stabilizer of four points in G contains no involution. Since G is 4-transitive on Ω, there exist involutions a and b such that

$$a=(1)(2)(3\ 4)\cdots, \qquad b=(1\ 2)(3)(4)\cdots.$$

Set $|ab|=2s$. Then s is odd, because $(ab)^2 \in G_{1234}$. Set $c=(ab)^s$. Then c is an involution with $ac=ca$. Since G is 4-transitive on Ω, G contains an element g such that $a^g=(1\ 2)(3\ 4)\cdots$. Then $a^g c=(1)(2)(3)(4)\cdots$. Hence a^g is conjugate to c, because $|a^g c|$ is odd. Thus c is conjugate to a, and $|F(c)|$ (=the number of the points left fixed by c) is two or three. Suppose that $|F(c)|=3$. We may assume that

$$c=(1\ 2)(3\ 4)(5)(6)(7)\cdots, \quad \text{and} \quad a=(1)(2)(3\ 4)(5\ 6)(7)\cdots.$$

We remark that $\langle a, c\rangle$ is semiregular on $\Omega-\{1,2,3,4,5,6,7\}$. Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be any orbit of $\langle a, c\rangle$ of length four. We may assume that

$$a=(1)(2)(3\ 4)(5\ 6)(7)(\alpha_1\ \alpha_2)(\alpha_3\ \alpha_4)\cdots,$$
$$c=(1\ 2)(3\ 4)(5)(6)(7)(\alpha_1\ \alpha_3)(\alpha_2\ \alpha_4)\cdots.$$

Since G is 4-transitive on Ω, G contains an element d of order four such that $d=(\alpha_1\alpha_2\alpha_3\alpha_4)\cdots$. Then, d^2 is an involution and $d^2c=(\alpha_1)(\alpha_2)(\alpha_3)(\alpha_4)\cdots$. Hence d^2 is conjugate to c, and there exists an element h in $G_{\alpha_1\alpha_2\alpha_3\alpha_4}$ such that $c=(d^2)^h=(d^h)^2$. Let us replace d^h with d. Then $d^2=c$, $d=(\alpha_1\alpha_2\alpha_3\alpha_4)\cdots$, and d fixes $\{5, 6, 7\}$ as a set. If $d^{\{5,6,7\}}$ is a transposition then $(ad)^3=(\alpha_1\ \alpha_3)(\alpha_2)(\alpha_4)(5)(6)(7)\cdots$. Therefore $G_{\alpha_2\alpha_4 56}$ contains an involution, a contradiction. Thus, we have

$$d=(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)(5)(6)(7)\cdots.$$

Since $dd^a \in G_{\alpha_1\alpha_2\alpha_3\alpha_4 56}=1$, we get $d^a=d^{-1}$. Hence $\langle a, d\rangle$ is a dihedral group of order eight. Since d normalizes $\langle a, c\rangle$, and $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$ are the orbits of $\langle a, c\rangle$ of length two, we have

$$d=(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)(1\ 3\ 2\ 4)(5)(6)(7)\cdots$$

or

$$d=(\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)(1\ 4\ 2\ 3)(5)(6)(7)\cdots.$$

Hence,

$$ad=(1\ 3)(2\ 4)(\alpha_1\ \alpha_3)(\alpha_2)(\alpha_4)(5\ 6)(7)\cdots$$

or

$$ad=(1\ 4)(2\ 3)(\alpha_1\ \alpha_3)(\alpha_2)(\alpha_4)(5\ 6)(7)\cdots.$$

Thus, we have the following result: For any orbit $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of $\langle a, c\rangle$ of length four, there exists an involution x in G such that $x^{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}$ is a transposition and $x=(1\ 3)(2\ 4)(5\ 6)(7)\cdots$ or $(1\ 4)(2\ 3)(5\ 6)(7)\cdots$. Since $\langle a, c\rangle$ has infinite number of orbits of length four and any involution fixes at most three points, we get infinite number of involutions of the form $(1\ 3)(2\ 4)(5\ 6)(7)\cdots$ or $(1\ 4)(2\ 3)(5\ 6)(7)\cdots$. Hence we have $|G_{1234567}|=\infty$, a contradiction.

If $|F(c)|=2$, then we get a contradiction by the similar argument to the case $|F(c)|=3$.

Thus, the stabilizer of four points in G contains an involution. Since G_{1234} is a Frobenius complement of the Frobenius group G_{123}, G_{1234} contains the unique involution by Lemma. Let i be the involution of G_{1234}. We may assume that

$$i=(1)(2)(3)(4)(5\ 6)\cdots.$$

Let $(\alpha\ \beta)$ be any transposition of i different from $(5\ 6)$. Then i normalizes $G_{56\alpha\beta}$, and i centralizes the unique involution x of $G_{56\alpha\beta}$, where $x=(1\ 2)(3\ 4)(5)(6)\cdots$, $(1\ 3)(2\ 4)(5)(6)\cdots$ or $(1\ 4)(2\ 3)(5)(6)\cdots$. Since i has infinite number of transpositions and any involution fixes at most four points, we get infinite number of involutions of the form $(1\ 2)(3\ 4)(5)(6)\cdots$, $(1\ 3)(2\ 4)(5)(6)\cdots$ or $(1\ 4)(2\ 3)(5)(6)\cdots$. Hence we have $|G_{123456}|=\infty$, a contradiction.

References

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