65. On a Construction of Multi-Soliton Solutions of the Pohlmeyer-Lund-Regge System and the Classical Massive Thirring Model

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1. This is a sequel to our previous paper [1]. We extend the method presented in [1] to the equation of the system of Pohlmeyer [11] and Lund-Regge [7]

$$\begin{aligned}&\alpha_{\xi\eta}-\beta_\xi\beta_\eta \sin(\alpha/2)/2\cos^3(\alpha/2)+\sin\alpha=0,\\&\beta_{\xi\eta}+(\alpha_\xi\beta_\eta+\alpha_\eta\beta_\xi)/\sin\alpha=0\end{aligned}\tag{1}$$

and the equation of the classical massive Thirring model

$$\begin{aligned}&iu_\eta+2v+2u|v|^2=0,\\&iv_\xi+2u+2v|u|^2=0.\end{aligned}\tag{2}$$

As is well known, a typical class of nonlinear equations solvable by the inverse scattering method is the Zakharov-Shabat equations, which are a class of equations for $u_j(x,y,t)$, $v_k(x,y,t)$ expressed as

$$[\textstyle\sum_{j=0}^n u_jD^j-\partial/\partial y,\ \sum_{k=0}^m v_kD^k-\partial/\partial t]=0,\quad D=\partial/\partial x.\tag{3}$$

The structures of this class are fairly well understood, especially in connection with algebraic geometry (see, for example, [4], [2], [9], [10]). However equations (1) and (2) are not included in this class. In [12], Zakharov and Mikhailov proposed that the sine-Gordon equation and equations (1), (2) are examples of the following class of equations. This class consists of equations of the relativistically invariant two-dimensional models in the classical field theories, which are expressed as the compatibility conditions of two linear differential equations

$$i\Phi_\xi=U(\xi,\eta,\lambda)\Phi,\quad i\Phi_\eta=V(\xi,\eta,\lambda)\Phi,\quad \Phi=\Phi(\xi,\eta,\lambda),\tag{4}$$

U and V being matrix-valued rational functions of complex parameter λ with poles independent of (ξ,η). For equations in this class, Zakharov-Mikhailov gave a method of constructing new solutions when a particular solution is given (a kind of Bäcklund transformation). Therefore equations in this class are considered to have properties in common and to be investigated as a whole. By now, the structures of this class are less investigated compared with the class of equations (3), though for specific equations in this class the inverse scattering method has been applied, for example, for (1) by Lund [8] and Kulish [5] and for (2) by Kuznetsov-Mikhailov [6] and Kaup-Newell [3].

Here we construct multi-soliton solutions of (1) and (2), by characterizing the corresponding simultaneous solutions of linear equations as in the case of the equations (3), with a hope that these calculations together with the calculations for the sine-Gordon equation in [1] may help to understand the structures of the class of nonlinear equations expressed as the compatibility conditions of (4).

Details will appear elsewhere.

2. Construction of solutions. The equation (1) is the compatibility condition of

$$
\begin{aligned}
&i\Phi_\xi+\begin{pmatrix}0 & a^*\\ a & 0\end{pmatrix}\Phi+2^{-1}\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi=0,\\
&i\Phi_\eta+2^{-1}\lambda^{-1}\begin{pmatrix}\cos\alpha & -\sin\alpha\exp(-i\omega)\\ -\sin\alpha\exp(i\omega) & -\cos\alpha\end{pmatrix}\Phi=0
\end{aligned}
\tag{5}
$$

where $\lambda\in C$, $a=i(\sin\alpha\exp(i\omega))_\xi/2\cos\alpha$, $\omega_\xi=\beta_\xi\cos\alpha/2\cos^2(\alpha/2)$, $\omega_\eta=\beta_\eta/2\cos^2(\alpha/2)$ and a^* being the complex conjugate of a ([11]).

The equation (2) is the compatibility condition of

$$
\begin{aligned}
&i\Phi_\xi+2\lambda\begin{pmatrix}0 & b\\ b^* & 0\end{pmatrix}\Phi+\lambda^2\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\Phi=0,\\
&i\Phi_\eta+2|v|^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi+2\lambda^{-1}\begin{pmatrix}0 & c\\ c^* & 0\end{pmatrix}\Phi+\lambda^{-2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\Phi=0
\end{aligned}
\tag{6}
$$

where $b=u\exp\left(2i\int_\xi^\infty|u|^2\,d\xi\right)$, $c=v\exp\left(2i\int_\xi^\infty|u|^2\,d\xi\right)$ ([3]).

We construct solutions of (1), (2) by constructing simultaneous solutions of linear equations of forms (5), (6).

We consider (2×2)-matrix-valued function $\Phi(\xi,\eta,\lambda)=(\Phi_{jk}(\xi,\eta,\lambda))$ $((\xi,\eta)\in\boldsymbol{R}^2,\lambda\in\boldsymbol{C})$ of the following forms:

$$
\begin{aligned}
&\Phi_{n1}(\xi,\eta,\lambda)=\left(\textstyle\sum_{j=0}^N\phi_{nj}(\xi,\eta)\lambda^j\right)\exp(2^{-1}i(\lambda\xi+\lambda^{-1}\eta)),\\
&\qquad\qquad n=1,2,\ \phi_{1N}\equiv1,\ \phi_{2N}\equiv0,\\
&\Phi_{12}(\xi,\eta,\lambda)=\Phi_{21}{}^*(\xi,\eta,\lambda^*),\ \Phi_{22}(\xi,\eta,\lambda)=-\Phi_{11}{}^*(\xi,\eta,\lambda^*)
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
&\Phi_{12}(\xi,\eta,\lambda)=\left(\textstyle\sum_{j=1}^N\phi_{1j}(\xi,\eta)\lambda^{2j-1}\right)\exp(i(\lambda^2\xi+\lambda^{-2}\eta)),\\
&\Phi_{22}(\xi,\eta,\lambda)=\left(\textstyle\sum_{j=0}^N\phi_{2j}(\xi,\eta)\lambda^{2j}\right)\exp(i(\lambda^2\xi+\lambda^{-2}\eta)),\ \phi_{20}\equiv1,\\
&\Phi_{11}(\xi,\eta,\lambda)=-\Phi_{22}{}^*(\xi,\eta,\lambda^*),\quad \Phi_{21}(\xi,\eta,\lambda)=\Phi_{12}{}^*(\xi,\eta,\lambda^*)
\end{aligned}
\tag{8}
$$

where N being an arbitrary positive integer.

Then as in [1], by the the conditions

$$\Phi_{n1}(\xi,\eta,\alpha_j)=c_j\Phi_{n2}(\xi,\eta,\alpha_j),\qquad j=1,\cdots,N,\ n=1,2,$$

the function $\Phi(\xi,\eta,\lambda)$ of the form (7) (resp. (8)) is uniquely determined, where $\alpha_1,\cdots,\alpha_N$ being complex numbers such that $\alpha_j\neq\alpha_k$ (resp. $\alpha_j^2\neq\alpha_k^2$) $(j\neq k)$ and for any $\{j_1,\cdots,j_k\}\subset\{1,\cdots,N\}$ $\alpha_{j_1}+\cdots+\alpha_{j_k}\notin\boldsymbol{R}$ (resp. $\alpha_{j_1}^2+\cdots+\alpha_{j_k}^2\notin\boldsymbol{R}$), and $c_1,\cdots,c_N$ being nonzero complex numbers. In particular, functions $\phi_{nj}(\xi,\eta)$ $(n=1,2,j=0,1,\cdots,N)$ are rational functions of $\exp(2^{-1}i(\alpha_k\xi+\alpha_k^{-1}\eta))$, $\exp(2^{-1}i(\alpha_k^*\xi+\alpha_k^{*-1}\eta))$ (resp. $\exp(i(\alpha_k^2\xi+\alpha_k^{-2}\eta))$, $\exp(i(\alpha_k^{*2}\xi+\alpha_k^{*-2}\eta))$), $k=1,\cdots,N$.

Further the function $\Phi(\xi, \eta, \lambda)$ determined above satisfies the equation (5) (resp. (6)) with coefficients

$$a=\phi_{2,N-1}, \quad \cos\alpha=(|\phi_{10}|^2-|\phi_{20}|^2)/(|\phi_{10}|^2+|\phi_{20}|^2),$$
$$\sin\alpha \exp(i\omega)=-2\phi_{10}{}^*\phi_{20}/(|\phi_{10}|^2+|\phi_{20}|^2)$$

(resp. $a=\phi_{1N}/\phi_{2N}$, $b=\phi_{11}$).

Finally by using equations (5), (6) with coefficients determined above, we have

Theorem. i) *The pair of functions*

$$\alpha=\arccos\{(|\phi_{10}|^2-|\phi_{20}|^2)/(|\phi_{10}|^2+|\phi_{20}|^2)\},$$
$$\beta=-i\log(\phi_{20}/\phi_{20}{}^*)+\beta_0, \qquad \beta_0\in \boldsymbol{R}$$

is the solution of (1),

ii) *the pair of functions*

$$u(\xi,\eta)=\phi_{1N}(\xi,\eta)/\phi_{2N}(\infty,\eta), \qquad v(\xi,\eta)=\phi_{11}(\xi,\eta)\phi_{2N}(\xi,\eta)/\phi_{2N}(\infty,\eta)$$

is the solution of (2).

These solutions we have constructed are determined by $2N$ constants α_j, c_j and are expressed by $\phi_{nj}(\xi,\eta)$ which are rational functions of exponential of linear functions of ξ, η, as we noted above. For example, for $N=1$, $\alpha_1=d\exp(i\delta)$, $c_1=r\exp(i\gamma)$, we have

i) $\alpha=2\arcsin[\sin\delta/\cosh\{\sin\delta(d\xi-d^{-1}\eta)+\log r\}]$,

$$\beta=-\cos\delta(d\xi+d^{-1}\eta)+\beta_0$$

which is the same as the one-soliton solution of (1) given by Lund [8], in view of the relations

$$\theta=2^{-1}\alpha, \quad \lambda_\xi=2^{-1}\beta_\xi\tan^2(\alpha/2), \quad \lambda_\eta=-2^{-1}\beta_\eta\tan^2(\alpha/2)$$

where θ, λ being the variables in [8].

ii) $u=-id\sin(2\delta)\exp\{-2i\cos(2\delta)(d^2\xi+d^{-2}\eta)-i\gamma\}\operatorname{sech}\{2\sin(2\delta)$
$\times(d^2\xi-d^{-2}\eta)-\log r+i\delta\}$,

$v=id^{-1}\sin(2\delta)\exp\{-2i\cos(2\delta)(d^2\xi+d^{-2}\eta)-i\gamma\}\operatorname{sech}\{2\sin(2\delta)$
$\times(d^2\xi-d^{-2}\eta)-\log r-i\delta\}$

which is the same as the one-soliton solution of (2) given by Kuznetsov-Mikhailov [6] and Kaup-Newell [3].

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