85. On a Diophantine Equation

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(Communicated by Kunihiko KODAIRA, M. J. A., Nov. 12, 1979)

The purpose of this note is to prove the following

Theorem. *The only integer solutions of the Diophantine equation*

$$3y^2=x^3+2x \tag{1}$$

are given by $x=0, 1, 2$ *and* 24.

By a classical theorem of A. Thue on the elliptic Diophantine equation we know that the equation (1) has only finitely many solutions in integers x and y.*) In order to effectively determine all the solutions of (1), we shall make use of some results due to W. Ljunggren [1], [2], and [3].

We write the equation (1) in the form

$$y^2=\frac{1}{3}x(x^2+2)$$

and distinguish three cases according as $x\equiv 0, 1$ or $2 \pmod 3$.

Solutions with $x\equiv 0 \pmod 3$. Write $x=3x_1$. We have then $y^2=x_1\cdot(9x_1^2+2)$, where $d_1=\text{g.c.d.}\ (x_1, 9x_1^2+2)=1$ or 2.

If x_1 is an odd integer, then $d_1=1$ and we have $x_1=Y^2$, $9x_1^2+2=X^2$ for some integers X, Y with g.c.d. $(X, Y)=1$. Eliminating x_1 from these equations, we get $X^2-9Y^4=2$; but this equation has no integer solutions X, Y, since the congruence $X^2\equiv 2 \pmod 3$ is insoluble.

If x_1 is an even integer, then $d_1=2$ and so $x_1=2Y^2$, $9x_1^2+2=2X^2$ for some integers X, Y with g.c.d. $(X, Y)=1$. Eliminating x_1, we get the equation

$$X^2-18Y^4=1. \tag{2}$$

which can be rewritten in the form $X^2-2(3Y^2)^2=1$.

Now, the solutions in non-negative integers u, v of the equation

$$u^2-2v^2=1$$

are given by $u=u_{2m}$, $v=v_{2m}$ $(m=0, 1, 2, \cdots)$, where

$$u_n+\sqrt{2}\,v_n=(1+\sqrt{2})^n \qquad (n=0, 1, 2, \cdots).$$

The sequences u_n, v_n are determined by the relations

$$u_0=1, \quad u_1=1, \quad u_{n+1}=2u_n+u_{n-1} \qquad (n\geqq 1),$$
$$v_0=0, \quad v_1=1, \quad v_{n+1}=2v_n+v_{n-1} \qquad (n\geqq 1).$$

Lemma 1. *We have for all* $m\geqq 0$

*) In fact, the equation (1) arises from a problem concerning MacMahon's 'chromatic' triangles in graph theory and, according to M. Gardner, it is known that the only solutions of (1) with $x\leqq 5{,}000$ are as listed in the theorem.

$$\text{g.c.d.}\ (u_m, v_m)=\text{g.c.d.}\ (u_m, u_{2m})=\text{g.c.d.}\ (u_{2m}, v_m)=1.$$

Proof will be easily carried out by noticing the relations

(3) $$u_n^2-2v_n^2=(-1)^n \qquad (n\geqq 0)$$

and

(4) $$u_{2n}=u_n^2+2v_n^2 \qquad (n\geqq 0)$$

which is a special case of

(5) $$u_{m+n}=u_m u_n+2v_m v_n \qquad (m, n\geqq 0).$$

Lemma 2. *We have*

$u_n\equiv 0 \pmod 3$ *if and only if* $n\equiv 2 \pmod 4$

and

$v_n\equiv 0 \pmod 3$ *if and only if* $n\equiv 0 \pmod 4$.

Proof. Indeed, we observe that

$n\equiv 0$	1	2	3	4	5	6	7	(mod 8)
$u_n\equiv 1$	1	0	1	2	2	0	2	(mod 3)
$v_n\equiv 0$	1	2	2	0	2	1	1	(mod 3).

This can be readily verified by making use of the defining relations for u_n and v_n, or of the relations (5) and

(6) $$v_{m+n}=u_n v_m+u_m v_n \qquad (m, n\geqq 0).$$

Now suppose that we have $v_{4m}=3Y^2$ $(m\geqq 0)$ for some integer Y. Here $v_{4m}=4u_m u_{2m} v_m$ since we have, by (6), $v_{2n}=2u_n v_n$ for all n.

Case 1. $m\equiv 0 \pmod 4$. In this case v_m is a multiple of 3 by Lemma 2, and we have by Lemma 1

$$u_m=r^2, \quad u_{2m}=s^2, \quad v_m=3t^2$$

for some non-negative integers r, s, t with $2rst=Y$. Putting these into the relations (3) and (4) (both with $n=m$) gives

$$r^4-18t^4=1 \quad \text{and} \quad s^2=r^4+18t^4.$$

Eliminating t from these equations, we thus otain the equation

(7) $$s^2=2r^4-1.$$

W. Ljunggren [2, § 2] has proved that the only solutions in positive integers (or, equivalently, non-negative integers) r, s of the equation (7) are

$$(r, s)=(1, 1) \quad \text{and} \quad (13, 239);$$

the former of these will give $t=0$, so that $v_m=0$, $m=0$, $Y=0$ and hence $x=0$, and the latter does not satisfy our requirement and there are no corresponding solutions x.

Case 2. $m\equiv 2 \pmod 4$. By Lemma 2 u_m is then divisible by 3 and we have, by Lemma 1 again,

$$u_m=3r^2, \quad u_{2m}=s^2, \quad v_m=t^2$$

for some positive integers r, s, t with $2rst=Y$. We have, by (4) (with $n=m$), $s^2=9r^4+2t^4$, which is obviously impossible, since g.c.d. $(t, 3)=1$ by Lemma 1, and 2 is a (uniques) quadratic non-residue (mod 3).

Case 3. $m\equiv 1 \pmod 2$. In this case u_{2m} is a multiple of 3 by Lemma 2, and we have, by Lemma 1,

$$u_m=r^2, \quad u_{2m}=3s^2, \quad v_m=t^2$$

for some positive integers r, s, t with $2rst=Y$. The relations (3) and (4) (with $n=m$) will yield the equations

$$r^4-2t^4=-1 \quad \text{and} \quad 3s^2=r^4+2t^4,$$

whence

$$3s^2-2r^4=1. \tag{8}$$

By a theorem of Ljunggren [1, Satz 3] the equation (8) has at most one solution in positive integers r, s; hence, $r=s=1$ is the unique positive solution of (8), giving $t=1, u_m=v_m=1$ and so $m=1$. Hence we have $v_{4m}=v_4=12$, $x=6Y^2=2v_4=24$.

Solutions with $x\equiv 1 \pmod 3$. Write $x=3x_1+1$. Then we have $y^2=(3x_1+1)(3x_1^2+2x_1+1)$, where $d_2=\text{g.c.d.}\ (3x_1+1, 3x_1^2+2x_1+1)=1$ or 2.

If $3x_1+1$ is odd, then $d_2=1$ and we have $3x_1+1=Y^2$, $3x_1^2+2x_1+1=X^2$ for some integers X, Y with g.c.d. $(X, Y)=1$, and elimination of x_1 will yield the equation

$$3X^2-Y^4=2. \tag{9}$$

This equation has an obvious solution $X=Y=1$, and we find by applying a theorem of Ljunggren [3, Satz II] that $X=Y=1$ is the unique positive solution of (9), and this gives the solution $x=Y^2=1$ of the equation (1).

If $3x_1+1$ is even, then $d_2=2$ and we have $3x_1+1=2Y^2$, $3x_1^2+2x_1+1=2X^2$ for some integers X, Y with g.c.d. $(X, Y)=1$; but this is impossible since the congruence $2Y^2\equiv 1 \pmod 3$ has no solutions in Y.

Solutions with $x\equiv 2 \pmod 3$. Put $x=3x_1-1$. Then we have $y^2=(3x_1-1)(3x_1^2-2x_1+1)$, where g.c.d. $(3x_1-1, 3x_1^2-2x_1+1)=1$ or 2.

Since $3x_1-1=Y^2$ is impossible in integers x_1, Y, we must have $3x_1-1$ even, and so $3x_1-1=2Y^2$, $3x_1^2-2x_1+1=2X^2$ for some integers X, Y with g.c.d. $(X, Y)=1$, whence

$$3X^2-2Y^4=1. \tag{10}$$

The equation (10), which is satisfied by $X=Y=1$, has at most one solution in positive integers X and Y, again by Ljunggren's [3, Satz II]. Hence, $X=Y=1$ is the unique positive solution of (10), and so $x=2Y^2=2$ is the only integer solution of the equation (1) with $x\equiv 2 \pmod 3$.

The proof of our theorem is now complete.

References

[1] W. Ljunggren: Über die unbestimmte Gleichung $Ax^2-By^4=C$. Archiv for Math. og Naturvid. (oslo), **41**, nr. 10 (1938).

[2] ——: Zur Theorie der Gleichung $x^2+1=Dy^4$. Avh. det Norske Vid.-Akad. Oslo. I. Mat.-Naturvid. Klasse, nr. 5 (1942).

[3] ——: Ein Satz über die diophantische Gleichung $Ax^2-By^4=C\,(C=1, 2, 4)$. Tolfte Skandinaviska Matematikerkongressen i Lund (1953), pp. 188–194.