75. *Studies on Holonomic Quantum Fields. XVI*

Density Matrix of Impenetrable Bose Gas

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In this article we report the following result concerning a system of impenetrable bosons in one dimension at zero temperature: The one particle reduced density matrix $\rho(x)$ satisfies a non-linear differential equation, an equivalent of a Painlevé equation of the fifth kind. This enables us to calculate the small and large x behaviors of $\rho(x)$ to an arbitrary order.

For the statement of the problem see [1] and references cited therein. As mentioned in [2], our calculation is done by relating the problem to the preceding result obtained there concerning the double scaling limit of the XY model.

Main results are summarized in §1. Their derivation is briefly described in §§2–3.

§1. Results. Let $\rho(|x-x'|)$ denote the thermodynamic limit of the one particle reduced density matrix with $\rho_0=\rho(0)$ normalized to be π^{-1} (for the definition see [1]). It is known ([3]) that $\rho(x)$ is an entire function of x.

We find that $\rho(x)$ is expressed as

$$\rho(x)=\rho_0 \exp\int_0^x dx'\left(\frac{x'}{4y(1-y)^2}\left(\left(\frac{dy}{dx'}\right)^2+4y^2\right)-\frac{(1+y)^2}{4x'y}\right) \tag{1}$$

with $y=y(x')$,

where $y=y(x)$ is a solution of the following Painlevé equation of the fifth kind:

$$\frac{d^2y}{dx^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2-\frac{1}{x}\frac{dy}{dx}+\frac{(y-1)^2}{x^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma y}{x}+\frac{\delta y(y+1)}{y-1} \tag{2}$$

with $\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$, $\gamma=-2i$, $\delta=2$.

If we set

$$\sigma(x)=x\frac{d}{dx}\log\rho(x), \tag{3}$$

then $\sigma=\sigma(x)$ itself satisfies the non-linear ordinary differential equation

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$$\left(x\frac{d^2\sigma}{dx^2}\right)^2=-4\left(x\frac{d\sigma}{dx}-1-\sigma\right)\left(x\frac{d\sigma}{dx}+\left(\frac{d\sigma}{dx}\right)^2-\sigma\right). \tag{4}$$

It is worth noting that the density matrix of free fermion $\rho_{F.F.}(x)=\frac{\sin x}{\pi x}$ also satisfies the same equations (3)–(4).

The small and large x behaviors of $\rho(x)$ are determined from the differential equation (4). The boundary condition is fixed by the first two terms obtained by Lenard [3] ($x\to 0$) and Vaidya-Tracy [1] ($x\to\infty$). The results read as follows.

$$\begin{aligned}\rho_0^{-1}\rho(x)=1&-\frac{1}{3!}x^2+\frac{2C}{3^2}x^3+\frac{1}{5!}x^4-\frac{11C}{3^3\cdot 5^2}x^5-\frac{1}{7!}x^6\\&+\frac{61C}{2^2\cdot 3^3\cdot 5^2\cdot 7^2}x^7+\left(\frac{1}{9!}+\frac{C^2}{3^5\cdot 5^2}\right)x^8-\frac{11\cdot 23C}{2^3\cdot 3^6\cdot 5^3\cdot 7^2}x^9\\&-\left(\frac{1}{11!}+\frac{163C^2}{2\cdot 3^5\cdot 5^4\cdot 7^2}\right)x^{10}+\cdots\qquad(x\to 0)\end{aligned} \tag{5}$$

with $C=\frac{1}{2\pi}$. $\left(\text{The choice } C=0 \text{ leads to } \rho_0^{-1}\rho_{F.F.}(x)=\frac{\sin x}{x}\right)$.

$$\begin{aligned}\rho_0^{-1}\rho(x)=\frac{\rho_\infty}{\sqrt{x}}\Bigg(1&+\frac{1}{2^3x^2}\left(\cos 2x-\frac{1}{4}\right)+\frac{3}{2^4x^3}\sin 2x\\&+\frac{3}{2^8x^4}\left(\frac{11}{2^3}-31\cdot\cos 2x\right)-\frac{3\cdot 151}{2^9x^5}\sin 2x\\&+\frac{3^2}{2^{14}x^6}\left(3\cdot 1579\cdot\cos 2x-\frac{17\cdot 19}{2^2}\right)\\&+\frac{3^3\cdot 5\cdot 7\cdot 311}{2^{15}x^7}\sin 2x+\frac{1}{2^{13}x^8}\\&\cdot\left(c_8-\frac{3^2\cdot 2064719}{2^6}\cos 2x+\frac{3^2}{2^2}\cos 4x\right)+\cdots\Bigg)\\&\qquad\qquad(x\to\infty)\end{aligned} \tag{6}$$

where c_8 is a constant. The constant ρ_∞ was determined by Vaidya-Tracy [1] to be $\pi e^{1/2}2^{-1/3}A^{-6}$ ($A=$Glaisher's constant).

Remark 1. Sometimes it is convenient to rewrite (4) as a first order system

$$\begin{cases}x\dfrac{d\xi}{dx}=(1+\xi^2)\sin\eta\\[2mm] x\dfrac{d\eta}{dx}=2x+2\xi(1+\cos\eta)\end{cases} \tag{7}$$

where ξ,η are related to σ and y through

$$\begin{aligned}&\xi=\frac{d\sigma}{dx},\qquad \eta=i\log(-y)\\&\sigma=x\xi+\frac{1}{2}(1+\xi^2)(1+\cos\eta)-1.\end{aligned} \tag{8}$$

Remark 2. The system (7) has $x=\infty$ as a singularity of "irregular Briot-Bouquet" type (see [4]). This guarantees the asymptotic structure at $x\to\infty$ $\sigma(x)=\sum_{n=0}^{\infty}\frac{\sigma_n(x)}{x^n}$, where $\sigma_n(x)$ are polynomials in $e^{\pm 2ix}$ (cf. [1]).

Remark 3. The results in [1] coincide with ours to the orders of x^7 (for small x) and $\frac{1}{x^2}$ (for large x). The higher order terms are, however, to be corrected as above so that the differential equation (4) holds.

§ 2. The results of § 1 are obtained by relating the problem to the double scaling limit of the XY model ([5][2]). It has been observed by Schultz [6] that the system of impenetrable bosons, once discretized on a finite lattice, is equivalent to an isotropic spin $\frac{1}{2}XY$ chain. Making use of this connection Vaidya-Tracy [1] (see [7] for details) studied the one particle reduced density matrix as a limit of the double-scaled 2 point function $\tau_2(a_1, a_2; g)$ of the latter; namely we have, in the notation of [2],

$$\rho(|a_1-a_2|)=\lim_{g\to 0}\sqrt{g}\,\tau_2(a_1, a_2; g). \tag{9}$$

For finite $g>0$ the logarithmic derivative of the n point function $d\log\tau_n$ is expressed in terms of a solution of non-linear total differential equations involving $2n\times 2n$ matrices ((35), (36), (38) in [2]). We begin by showing that for $n=2$ the size of the matrices is reducible to 2×2. In the sequel we employ the notations in [2] without mentioning further.

Define $v_k(p)=v_k(p, x; \lambda, g)(k=0,1,2,3)$ by the series

$$v_k(p)=\sum_{l\geq 0}{}'(i\lambda)^l\int_{C_-\times\cdots\times C_-}\cdots\int\frac{dp_1}{2\pi}\cdots\frac{dp_l}{2\pi}e^{-ix(p_1+\cdots+p_l)} \times\frac{1}{p+p_1}\frac{1}{p_1+p_2}\cdots\frac{1}{p_{l-1}+p_l}\omega_1^{\varepsilon}\omega_2^{-\varepsilon}\omega_3^{\varepsilon}\cdots\omega_l^{(-)^{k+1}\varepsilon} \tag{10}$$

where $-p$ is supposed to lie outside the contour C_- of p_1-integration. Here the sum $\sum'$ extends over even l $(k=0,2)$ or odd l $(k=1,3)$, and $\varepsilon=+1$ $(k=0,3)$, $=-1$ $(k=1,2)$. These functions are related to $w_{jj'}^{(i)\varepsilon\varepsilon'}(p)$ in [2] through

$$\begin{aligned} w_{11}^{(1)\varepsilon\varepsilon'}(p)&=e^{-ia_1p}(\delta_{1\varepsilon}\omega(p)v_2(-p)+\delta_{-1\varepsilon}\varepsilon'v_0(-p))\\ w_{21}^{(1)\varepsilon\varepsilon'}(p)&=e^{-ia_1p}(\delta_{1\varepsilon}\varepsilon'v_3(-p)+\delta_{-1\varepsilon}\omega(p)v_1(-p))\\ w_{12}^{(2)\varepsilon\varepsilon'}(p)&=e^{-ia_2p}(\delta_{1\varepsilon}\varepsilon'v_3(p)+\delta_{-1\varepsilon}\omega(p)v_1(p))\\ w_{22}^{(2)\varepsilon\varepsilon'}(p)&=e^{-ia_2p}(\delta_{1\varepsilon}\omega(p)v_2(p)+\delta_{-1\varepsilon}\varepsilon'v_0(p)) \end{aligned} \tag{11}$$

where we have made the identification $x=a_2-a_1>0$, $\lambda=\lambda_{12}=\lambda_{21}$. The choice $\lambda=1$ corresponds to the original problem. Now set

$$Y(p)=\begin{pmatrix} v_0(p) & v_1(-p)\\ v_3(p) & v_2(-p)\end{pmatrix}\begin{pmatrix}1 & \\ & e^{ixp}\omega(p)\end{pmatrix}. \tag{12}$$

Then $\det Y(p)=e^{ixp}\omega(p)$, so that $Y(p)$ is holomorphic and invertible everywhere except for $p=c_s$ ($s=1,2,3,4$) and $p=\infty$. At $p=c_s$ it has an expression

(13) $$Y(p)=\Phi_s(p)(p-c_s)^{L'_s} \qquad (s=1,2,3,4)$$

$$L'_1=L'_2=\begin{pmatrix} 0 & 0 \\ -\dfrac{\lambda}{2} & \dfrac{1}{2} \end{pmatrix},\quad L'_3=L'_4=\begin{pmatrix} 0 & \dfrac{\lambda}{2} \\ 0 & \dfrac{1}{2} \end{pmatrix}$$

with $\Phi_s(p)$ holomorphic and invertible at $p=c_s$. In particular $Y(p)$ has a monodromic property

(14) $$\gamma_s Y(p)=Y(p)M_s,$$

$$M_1=M_2=\begin{pmatrix} 1 & 0 \\ 2\lambda & -1 \end{pmatrix},\qquad M_3=M_4=\begin{pmatrix} 1 & -2\lambda \\ 0 & -1 \end{pmatrix}.$$

For $p\to\infty$ $Y(p)$ has an asymptotic expansion

(15) $$Y(p)=\left(1+\frac{Y_1}{p}+\frac{Y_2}{p^2}+\cdots\right)\begin{pmatrix} 1 & \\ & p^2e^{ixp} \end{pmatrix}$$

valid in the full sector $0\leqq \arg p \leqq 2\pi$, $|p|\gg 1$.

We thus conclude that the matrix of 1-forms $\Omega=dY\cdot Y^{-1}$ takes the form (35) [2], where $A_\infty=\begin{pmatrix} 0 & \\ & ix \end{pmatrix}$ and

(16) $$A_s=\begin{cases} \dfrac{1}{2}\begin{pmatrix} v_1(-c_s) \\ v_2(-c_s) \end{pmatrix}(-\tilde{v}_3(c_s)\ \tilde{v}_0(c_s)) & (s=1,2) \\ \dfrac{1}{2}\begin{pmatrix} \tilde{v}_1(-c_s) \\ \tilde{v}_2(-c_s) \end{pmatrix}(-v_3(c_s)\ v_0(c_s)) & (s=3,4) \end{cases}$$

(17) $$\Theta=\begin{pmatrix} & -iv_1^{(0)} \\ -iv_3^{(0)} & \end{pmatrix}dx.$$

In (16) $\tilde{v}_k(p)$ is defined as the series (10) where the contour of p_1 is modified to encircle $-p$ (Fig. 1),

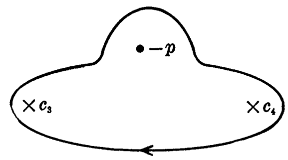

Fig. 1

and in (17) $v_k^{(0)}$ is defined by $v_k(p)=\sum_{n=0}^{\infty}\dfrac{v_k^{(n)}}{p^{n+1}}$ ($p\to\infty$) for $k=1,3$. With this replacement the deformation equations (36), (37) in [2] remain valid. We note that $A_{\infty 1}=\sum_{s=1}^{4} A_s$ is given by $A_{\infty 1}=\begin{pmatrix} 0 & -ixv_1^{(0)} \\ -ixv_3^{(0)} & 2 \end{pmatrix}$.

Finally the logarithmic derivative of the 2 point function $\omega=d\log\tau_2$ is given by

(18) $$\omega=-\mathrm{trace}\ Y_1dA_\infty+\sum_{s=1}^{4}\mathrm{trace}\ (\Phi_s(c_s)^{-1}\Phi'_s(c_s)L_sdc_s)-\frac{1}{8}\sum_{s\neq s'}\frac{d(c_s-c_{s'})}{c_s-c_{s'}}$$

$$=\mathrm{trace}\left(A_{\infty 2}dA_{\infty}+\frac{1}{2}\Theta A_{\infty 1}+\frac{1}{2}\sum_{s\neq s'}(A_sA_{s'}-L_s^2)d\log(c_s-c_{s'})\right.$$
$$\left.+\Gamma A_{\infty}\right)$$

with $L_s=\begin{pmatrix}0 & \\ & \frac{1}{2}\end{pmatrix}$, $A_{\infty 2}=\sum_{s=1}^4 A_sc_s$ and $\Gamma=\sum_{s=1}^4 A_s dc_s$. We have also

$$\frac{\langle\varphi^{(+)}(a_1)\varphi^{(+)}(a_2)\rangle}{\langle\varphi(a_1)\varphi(a_2)\rangle}=-iv_3^{(0)}=\frac{1}{x}(A_{\infty 1})_{21} \tag{19}$$
$$\frac{\langle\varphi^{(-)}(a_1)\varphi^{(-)}(a_2)\rangle}{\langle\varphi(a_1)\varphi(a_2)\rangle}=iv_1^{(0)}=-\frac{1}{x}(A_{\infty 1})_{12}.$$

§3. The limit $g\to 0$ corresponds to the pairwise coalescence of branch points $c_1, c_4\to 1$, $c_2, c_3\to -1$. Although the matrices A_s may diverge individually, the sums A_1+A_4, A_2+A_3 $\left(\text{and hence } \Theta=\left(\sum_{s=1}^4 A_s-\begin{pmatrix}0 & \\ & 2\end{pmatrix}\right)\frac{dx}{x}\right)$ tend to finite limits A_+, A_- respectively (cf. [8] §2.4). The limiting differential equations then read as

$$dY=\Omega Y \tag{20}$$
$$\Omega=\left(\frac{A_+}{p-1}+\frac{A_-}{p+1}+\begin{pmatrix}0 & \\ & ix\end{pmatrix}\right)dp+\begin{pmatrix}0 & \frac{1}{x}e_{12}\\ \frac{1}{x}e_{21} & ip\end{pmatrix}dx$$

where we have set

$$A_++A_-=\begin{pmatrix}0 & e_{12}\\ e_{21} & 2\end{pmatrix}. \tag{21}$$

Note that $\omega=d\log\tau_2$ (18) also remains finite, except for the divergent factor $-\frac{1}{8}\sum_{s\neq s'}\frac{d(c_s-c_{s'})}{c_s-c_{s'}}=d\log g^{-1/2}(1-g^2)^{-1/4}$ corresponding to (9).

Since the monodromy is preserved in the above process, the monodromy for (20) at $p=\pm 1$ is given by $M_4M_1=\begin{pmatrix}1-4\lambda^2 & 2\lambda\\ -2\lambda & 1\end{pmatrix}$ and $M_2M_3=\begin{pmatrix}1 & -2\lambda\\ 2\lambda & 1-4\lambda^2\end{pmatrix}=(M_4M_1)^{-1}$, respectively. On the other hand we have $A_+=A_-=\begin{pmatrix}0 & \\ & 1\end{pmatrix}$ for $\lambda=0$. Hence by continuity the eigenvalues of $A_\pm$ are either $(\theta(\lambda), 1-\theta(\lambda))$ or $(-\theta(\lambda), 1+\theta(\lambda))$ where $e^{\pm 2\pi i\theta(\lambda)}=1-2\lambda^2\pm 2i\lambda\sqrt{1-\lambda^2}$, $\theta(0)=0$. The result of Vaidya-Tracy [1] tells that the correct choice for $\lambda=1$ should be $\left(\frac{1}{2}, \frac{1}{2}\right)$ for both of $A_\pm$. Combining this with (21) we can set, for $\lambda=1$,

$$A_+=\frac{1}{2}\begin{pmatrix}i\xi & (1-i\xi)e^{-i(x+\chi)}\\ -(1-i\xi)e^{i(x+\chi)} & 2-i\xi\end{pmatrix} \tag{22}$$

$$A_- = \frac{1}{2}\begin{pmatrix} -i\xi & -(1+i\xi)e^{-i(x+\chi-\eta)} \\ (1+i\xi)e^{i(x+\chi-\eta)} & 2+i\xi \end{pmatrix}.$$

The fact that (20) has two regular singularities and an irregular singularity of rank one implies that the deformation equation $d\Omega-\Omega^2=0$ is integrable in terms of a Painlevé transcendent of the fifth kind. In our case, substituting (22) to

$$x\frac{dA_\pm}{dx} = \left[\begin{pmatrix} 0 & e_{12} \\ e_{21} & \pm ix \end{pmatrix}, A_\pm\right] \tag{23}$$

we obtain (7) and

$$x\frac{d\chi}{dx} = 2\xi\cos^2\left(\frac{\eta}{2}\right) + 2i\sin^2\left(\frac{\eta}{2}\right). \tag{24}$$

Finally (18) yields

$$\begin{aligned} \frac{d}{dx}\log\rho(x) &= i(A_+ - A_-)_{22} + \frac{1}{x}e_{12}e_{21} \\ &= \xi + \frac{1}{2x}(1+\xi^2)(1+\cos\eta) - \frac{1}{x}, \end{aligned} \tag{25}$$

which is the results (3) and (8).

Erratum and comment for XI [9].

p. 8, (37) [9] should be corrected as $\sigma[M]=\tau[T_M]^2$.

Theorem 5 has previously been obtained by Widom [10]. He has also shown [11] that the product of half infinite Toeplitz operator (32) [9] of M with that of M^{-1} differs from 1 by a trace-class operator, and that $\sigma[M]$ coincides with its determinant. The authors wish to thank Prof. H. Widom for calling their attention to his results.

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