97. *Congruences between Siegel Modular Forms of Degree Two*

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

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Introduction. We present some congruences between eigenvalues of Hecke operators on Siegel modular forms of degree two. In § 1, we give a congruence modulo 71^2 between two Siegel modular forms of degree two and weight 20 denoted here by $\chi_{20}^{(3)}$ and $[\Delta_{20}]$. This congruence seems to suggest that 71 will be an "exceptional prime" for $\chi_{20}^{(3)}$. In § 2, we give some other congruences. Some of them are reduced to the elliptic modular case by the recent results of Maass [5] and Andrianov [1]; Maass [5] proves, by using Shimura's theory of elliptic modular forms of half integral weight, that the Conjectures 1 and 2 of [4] hold (at least) for $k \leqq 20$. In § 3, we give a conjectural "interpretation" of these congruences.

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This paper may be considered as a supplement to [4]. The author would like to express his hearty thanks to Profs. J.-P. Serre and G. Shimura for their interests and encouragements in the early stage of [4]: §§ 1 and 2 of [4] were extracted from the author's letter to Prof. Shimura dated February 24, 1976, and Prof. Serre suggested the author in March of 1976 to study congruences between Siegel modular forms in connection with the Conjecture 1 and examples of [4].

§ 1. A congruence. We follow the notations of [4] throughout this paper. Let $\chi_{20}^{(3)} = f_{20} - g_{20} + 595200h_{20}$ with $f_{20} = 4\chi_{10}\varphi_4\varphi_6$, $g_{20} = 12\chi_{12}\varphi_4^2$, and $h_{20} = 48\chi_{10}^2$ as in § 5 of [4]. This is an eigen cusp form of degree two and weight 20. We denote by $[\Delta_{20}]$ the *eigen* modular form of degree two and weight 20 which is uniquely determined by $\Phi([\Delta_{20}]) = \Delta_{20}$, where $\Phi : M_{20}(\Gamma_2) \to M_{20}(\Gamma_1)$ is the Siegel Φ-operator. We prove the following congruence between $\chi_{20}^{(3)}$ and $[\Delta_{20}]$.

Theorem 1. $\lambda(m, \chi_{20}^{(3)}) \equiv \lambda(m, [\Delta_{20}]) \bmod 71^2$ *for all integers* $m \geqq 1$.

Proof. Let $e_{20} = 2^{-6} \cdot 3^{-3} \cdot (\varphi_4^5 - \varphi_4^2\varphi_6^2)$, then we have the following equality:

$$11[\Delta_{20}] + 38 \cdot 71^{-2}\chi_{20}^{(3)} = 11e_{20} - 922f_{20} - 614g_{20} - 5030400h_{20}.$$

This equality is proved as follows. Since $\Phi([\Delta_{20}]) = \Phi(e_{20}) = \Delta_{20}$ (hence $[\Delta_{20}]$

$-e_{20}$ is a cusp form), we can write $[\Delta_{20}]=e_{20}+\alpha f_{20}+\beta g_{20}+\gamma h_{20}$ with α, β, $\gamma \in \boldsymbol{C}$. By calculating the Fourier coefficients of $e_{20}, f_{20}, g_{20}, h_{20}$ at $(1,1,1)$, $(1,1,0)$, $(2,2,2)$, $(2,2,0)$, $(3,3,3)$, and using $a(2;(1,1,1))=a(2,2,2)$, $a(2;(1,1,0))=a(2,2,0)+2^{18}a(1,1,0)$, and $a(3;(1,1,1))=a(3,3,3)+3^{18}a(1,1,1)$, we have the following table.

	$a(1,1,1)$	$a(1,1,0)$	$a(2;(1,1,1))$	$a(2;(1,1,0))$	$a(3;(1,1,1))$
e_{20}	-28	726	-1000416	300164400	-14774647968
f_{20}	-1	2	-59616	336960	-1062044568
g_{20}	1	10	44064	4322880	831750552
h_{20}	0	0	3	18	2016

We have $\lambda(p,[\Delta_{20}])=(1+p^{18})\lambda(p,\Delta_{20})=119538120$ (resp. 19623622659480) for $p=2$ (resp. 3). (Note that $H_p(T,[\Delta_{20}])=H_p(T,\Delta_{20})H_p(p^{18}T,\Delta_{20})$.) In the equality $\lambda(m,[\Delta_{20}])[\Delta_{20}]=T(m)[\Delta_{20}]$, by comparing the Fourier coefficients at $(1,1,1)$ for $m=2$ and 3, and at $(1,1,0)$ for $m=2$, we have the following three equalities from the above table:

$$119538120(-28-\alpha+\beta)=-1000416-59616\alpha+44064\beta+3\gamma,$$
$$119538120(726+2\alpha+10\beta)=300164400+336960\alpha+4322880\beta+18\gamma,$$
$$19623622659480(-28-\alpha+\beta)$$
$$=-14774647968-1062044568\alpha+831750552\beta+2016\gamma.$$

Hence we have $\alpha=-4647840\cdot 71^{-2}\cdot 11^{-1}$, $\beta=-3095136\cdot 71^{-2}\cdot 11^{-1}$, and $\gamma=-25380864000\cdot 71^{-2}\cdot 11^{-1}$. Hence noting $4647840=922\cdot 71^2+38$, $3095136=614\cdot 71^2-38$, and $25380864000=5030400\cdot 71^2+595200\cdot 38$, we have the required result.

We prepare a lemma (whose analogue in the elliptic modular case may be well-known to specialists; cf. § 2 of Serre [6]). For each subring R of $\boldsymbol{C}$ and for integers $n\geqq 1$ and $k\geqq 0$, we put

$$M_k(\Gamma_n)_R=\{f\in M_k(\Gamma_n)\,|\,a(T,f)\in R \text{ for all } T\geqq 0\}$$

(an R-module) and $M(\Gamma_n)_R=\oplus_{k\geqq 0} M_k(\Gamma_n)_R$ (a graded R-algebra). For each f in $M_k(\Gamma_2)_R$, it follows from the explicit expression of the Fourier coefficients of $T(m)f$ using the Fourier coefficients of f that $T(m)f$ belongs to $M_k(\Gamma_2)_R$ and that $T(m): M_k(\Gamma_2)_R\to M_k(\Gamma_2)_R$ is an R-linear operator for integers $m\geqq 1$ and $k\geqq 4$. In [3], Igusa determined the structure of $M(\Gamma_2)_{\boldsymbol{Z}}$.

Lemma 1. *Let f and g be eigen modular forms in $M_k(\Gamma_2)$ $(k\geqq 4)$ and we assume the following three conditions.*

(1) *$\lambda(m,f)$ and $\lambda(m,g)$ are integers for all $m\geqq 1$.*

(2) *There exist co-prime non-zero integers b and c such that $h=f-b\cdot c^{-1}\cdot g$ belongs to $M_k(\Gamma_2)_{\boldsymbol{Z}}$.*

(3) *A Fourier coefficient of g is 1.*

Then: $\lambda(m, f) \equiv \lambda(m, g) \bmod c$ *for all integers* $m \geqq 1$.

Proof of Lemma 1. From the equality in (2), we have the following equality for the Fourier coefficients at T:

$$(*) \qquad a(T, h) = a(T, f) - b \cdot c^{-1} \cdot a(T, g).$$

On the other hand by applying the Hecke operator $T(m)$ to the equality in (2), we have: $T(m)h = \lambda(m, f)f - b \cdot c^{-1} \cdot \lambda(m, g)g$. Hence we have:

$$(**) \qquad a(m; T, h) = \lambda(m, f)a(T, f) - b \cdot c^{-1} \cdot \lambda(m, g)a(T, g).$$

From $(*)$ and $(**)$ we have:

$$(***) \quad \lambda(m, f)a(T, h) - a(m; T, h) = -b \cdot c^{-1}(\lambda(m, f) - \lambda(m, g))a(T, g).$$

By (1) and (2), $\lambda(m, f)h$ and $T(m)h$ belong to $M_k(\Gamma_2)_{\boldsymbol{Z}}$, hence the left hand side of $(***)$ is an integer. In $(***)$ taking T such that $a(T, g) = 1$ by (3), we have that $-b \cdot c^{-1}(\lambda(m, f) - \lambda(m, g))$ is an integer. Since b and c are co-prime by (2) we have: $\lambda(m, f) \equiv \lambda(m, g) \bmod c$.

(q.e.d. of Lemma 1)

We have Theorem 1 by applying Lemma 1 with $f = 11[\Delta_{20}]$, $g = -2^{-1} \cdot \chi_{20}^{(3)}$, $b = 76$, and $c = 71^2$. In fact, $a(T, g) = 1$ for $T = (1, 1, 1)$, and e_{20}, f_{20}, g_{20}, and h_{20} belong to $M_{20}(\Gamma_2)_{\boldsymbol{Z}}$ by Igusa [3], since $e_{20} = (Y_{12} - 144X_{12})X_4^2$, $f_{20} = -X_{10}X_4X_6$, $g_{20} = X_{12}X_4^2$, and $h_{20} = 3X_{10}^2$ in the notation of [3]. We note that $g = X_{10}X_4X_6 + 6X_{16}X_4 - 892800X_{10}^2$ belongs to $M_{20}(\Gamma_2)_{\boldsymbol{Z}}$ by [3] and that $\lambda(m, \chi_{20}^{(3)}) = a((1, 1, 1), T(m)g)$ is an integer for each $m \geqq 1$.

Q.E.D.

Theorem 1 seems to suggest that 71 will be an "exceptional prime" for $\chi_{20}^{(3)}$, if we assume the existence of conjectural ℓ-adic representations attached to $\chi_{20}^{(3)}$; cf. Deligne [2], Serre [6] and [7], and Swinnerton-Dyer [8] for the elliptic modular case. For example, assume the existence of an (irreducible) ℓ-adic representation $\rho_\ell : \mathrm{Gal}\,(K_\ell/\boldsymbol{Q}) \to GL(4, \boldsymbol{Z}_\ell)$ satisfying

$$\det(1 - T \cdot \rho_\ell(\mathrm{Frob}\,(p))) = H_p(T, \chi_{20}^{(3)})$$

for all primes $p \neq \ell$ (with usual notations as in the above papers), then it seems that

$$\mathrm{Image}\,(\tilde{\rho}_\ell) \subset \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \middle| A, B, D : 2 \times 2 \text{ matrices} \right\}$$

(up to conjugation) for $\ell = 71$ (or, more vaguely, Image $(\tilde{\rho}_\ell)$ is not "large" for $\ell = 71$). Here $\tilde{\rho}_\ell$ denotes the reduction modulo ℓ of ρ_ℓ. Let σ_ℓ be the 2-dimensional ℓ-adic representation attached to Δ_{20}, and χ_ℓ be the ℓ-adic cyclotomic representation. Then it seems moreover that:

$$\tilde{\rho}_\ell \cong \begin{pmatrix} \tilde{\sigma}_\ell & * \\ 0 & \tilde{\chi}_\ell^{18} \otimes \tilde{\sigma}_\ell \end{pmatrix} \qquad \text{for } \ell = 71.$$

Here $\tilde{\sigma}_\ell$ and $\tilde{\chi}_\ell$ denote the reduction modulo ℓ of σ_ℓ and χ_ℓ respectively. We note that Theorem 1 says that $H_p(T, \chi_{20}^{(3)}) \equiv H_p(T, [\Delta_{20}]) \bmod 71^2$ for all primes p. (We may consider $\sim$ as the reduction modulo ℓ^2.)

Remark 1. From the values in [4] we have: $\lambda(m, \chi_{20}^{(3)}) - \lambda(m, [\Delta_{20}]) = -71^2 \cdot 2^3 \cdot 3 \cdot 5 \cdot 199,\ -71^2 \cdot 2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 16091,\ 71^2 \cdot 2^6 \cdot 3^2 \cdot 5 \cdot 1497429851$, and

$-71^2 \cdot 2^{10} \cdot 3^6 \cdot 5 \cdot 23 \cdot 486735888236971$ for $m=2, 3, 4$, and 9 respectively. These values are sufficient to prove the congruence in Theorem 1 for $m=2^\mu \cdot 3^\nu$ with $\mu, \nu=0, 1, 2, \cdots$.

Remark 2. $[\Delta_{20}]$ is the Eisenstein series attached to Δ_{20} in the sense of R. P. Langlands and H. Klingen. We note two Fourier coefficients of $[\Delta_{20}]$: $a(1,1,1)=71^{-2} \cdot 11^{-1} \cdot 2^2 \cdot 19$, and $a(1,1,0)=71^{-2} \cdot 11^{-1} \cdot 2 \cdot 3^2 \cdot 577$.

Remark 3. Let $S_k^{\mathrm{I}}(\Gamma_2)$ be as in §4 of [4], and let $S_k^{\mathrm{II}}(\Gamma_2)$ be the orthogonal complement of $S_k^{\mathrm{I}}(\Gamma_2)$ in $S_k(\Gamma_2)$ with respect to the Petersson inner product. Then $\chi_{20}^{(3)}$ is the "first" example of an eigen cusp form in $S_k^{\mathrm{II}}(\Gamma_2)$: $S_{20}^{\mathrm{II}}(\Gamma_2)=\boldsymbol{C} \cdot \chi_{20}^{(3)}$ and $S_{20}^{\mathrm{I}}(\Gamma_2)=\boldsymbol{C} \cdot \chi_{20}^{(1)} \oplus \boldsymbol{C} \cdot \chi_{20}^{(2)}$.

§2. Some congruences. Theorem 2. *The following congruences hold for all integers* $m \geqq 1$:

(1) $\lambda(m, \chi_{10}) \equiv \lambda(m, \varphi_{10}) \bmod 43867$,

(2) $\lambda(m, \chi_{12}) \equiv \lambda(m, \varphi_{12}) \bmod 131 \cdot 593$,

(3) $\lambda(m, \chi_{14}) \equiv \lambda(m, \varphi_{14}) \bmod 657931$.

These congruences are proved by applying Lemma 1 to the following equalities:

$$\varphi_{10}-43867^{-1}2^{12}3^5 5^2 7 \cdot 53 \chi_{10}=\varphi_4\varphi_6,$$
$$691\varphi_{12}+131^{-1}593^{-1}2^{13}3^7 5^3 7^2 337\chi_{12}=441\varphi_4^3+250\varphi_6^2,$$
$$\varphi_{14}-657931^{-1}2^{14}3^5 5^2 7 \cdot 269\chi_{14}=\varphi_4^2\varphi_6.$$

The first two equalities are due to Igusa [12] and the third equality follows from Maass [14] (or as in the case of $[\Delta_{20}]$ in §1). Theorem 2 is extended to the case of congruence between an eigen cusp form in $S_k^{\mathrm{I}}(\Gamma_2)$ and φ_k for each even integer $k \geqq 10$ by the recent results of Maass [5] and Andrianov [1]; such congruences are reduced to the elliptic modular case: Cf. Serre [6] [7], Swinnerton-Dyer [8], Doi-Miyake [11], and Koike [13].

We note another congruence for χ_{12}. Let $[\Delta_{12}]$ be the eigen modular form in $M_{12}(\Gamma_2)$ uniquely determined by $\Phi([\Delta_{12}])=\Delta_{12}$, where $\Phi: M_{12}(\Gamma_2) \to M_{12}(\Gamma_1)$ is the Siegel Φ-operator.

Theorem 3. $\lambda(m, \chi_{12}) \equiv \lambda(m, [\Delta_{12}]) \bmod 7$ *for all integers* $m \geqq 1$.

This congruence is proved by applying Lemma 1 to the following equality: $[\Delta_{12}]-7^{-1}2^7 3^3\chi_{12}=e_{12}$, where $e_{12}=2^{-6}3^{-3}(\varphi_4^3-\varphi_6^2)=Y_{12}-144X_{12}$ belongs to $M_{12}(\Gamma_2)_{\boldsymbol{Z}}$ by Igusa [3].

§3. An interpretation. We give a conjectural "interpretation" of the primes appearing in the above congruences. This is motivated by the recent results of Doi-Hida [10]; they give much more precise interpretations of primes appearing in congruences between Hilbert modular forms by using $L_2^*(k, f)$ (see below). The author is grateful to Profs. K. Doi and M. Koike for communicating [10].

For each eigen cusp form f in $S_k(\Gamma_1)$, let $L_2(s, f)=L(s, f, \mathrm{Sym}^2)$ be the "second" L-function attached to f (Serre [6]; cf. [10], [16], [17]), and we put: $L_2^*(s,f)=L_2(s,f)(2\pi)^{-(2s-k+2)}\Gamma(s)\langle f,f\rangle^{-1}$. Here $\langle\ ,\ \rangle$ is the Petersson inner product normalized as in Shimura [15]. Then, by Sturm [16] and Zagier [17], we see that $L_2^*(s,f)$ is an algebraic number in $K_f=\boldsymbol{Q}(\{\lambda(m,f)\,|\,m\geqq 1\})$ for each even integer s with $k\leqq s\leqq 2k-2$. It turns out that $L_2^*(2k-2,f)$ is interesting for our purpose here. In fact, by a method of Zagier [17, Theorem 2], we have:

$$L_2^*(38,\Delta_{20})=71^2\cdot 7^2\cdot 11\cdot 2^8\cdot(3^2\cdot 5\cdot 283\cdot 617)^{-1}.$$

Hence 71^2 appears in the numerator of the rational number $L_2^*(38,\Delta_{20})$. (Cf. Remark 2) In the calculation of the above value, we use the following values of the arithmetical function $H(r,N)$ defined by Cohen [9]: $H(19,0)=-154210205991661/12$, $H(19,3)=-17612343854/3$, and $H(19,4)=-2404879675441/2$.

For each eigen cusp form f in $S_k(\Gamma_1)$, let $[f]$ be an eigen modular form in $M_k(\Gamma_2)$ satisfying $\Phi([f])=f$, where $\Phi: M_k(\Gamma_2)\to M_k(\Gamma_1)$ is the Siegel Φ-operator. Then the above fact seems to suggest the following: For a "certain" prime ideal $\mathfrak{l}$ (in $\bar{\boldsymbol{Q}}$) dividing the numerator of $L_2^*(2k-2,f)$ (and only for such $\mathfrak{l}$) there will exist an eigen cusp form F in $S_k(\Gamma_2)$ satisfying $\lambda(m,F)\equiv\lambda(m,[f]) \bmod \mathfrak{l}$ for all $m\geqq 1$. We note that the congruence in Theorem 3 also fits this, in fact $L_2^*(22,\Delta_{12})=7\cdot 2^8\cdot(3\cdot 23\cdot 691)^{-1}$ by Zagier [17]. We may note that the congruences in Theorem 2 are also considered as examples of this type with some modifications; note that $\Phi(\varphi_k)=E_k$ and $L_2(s,E_k)=\zeta(s)\zeta(s-k+1)\zeta(s-2k+2)$, so $L_2(2k-2,E_k)=-\zeta(2k-2)\zeta(k-1)/2$. In certain cases of modular forms for congruence subgroups, $L_2^*(2k-2,f)$ is essentially the Hurwitz number H_{2k-2}^{2k-2} (cf. Doi-Hida [10]).

The above "interpretation" seems to suggest vaguely the following: A prime (or prime ideal) giving a congruence between an Eisenstein series and a cusp form on a reductive group over a number field may be characterized by certain special values of (a system of) L-functions attached to the cusp form defining the Eisenstein series. Here automorphic forms (or automorphic representations) and L-functions are in the sense of R. P. Langlands. (Note that the L-function attached to the cusp form defining φ_k is $\zeta(s)$, so Theorem 2 also fits this well.)

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