89. On Some Series of Regular Irreducible Prehomogeneous Vector Spaces

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Let $\boldsymbol{Q}=\boldsymbol{C}\cdot 1+\boldsymbol{C}\cdot \boldsymbol{e}_1+\boldsymbol{C}\cdot \boldsymbol{e}_2+\boldsymbol{C}\cdot \boldsymbol{e}_1\boldsymbol{e}_2$ be the quaternion algebra over $\boldsymbol{C}$ defined by $\boldsymbol{e}_1^2=\boldsymbol{e}_2^2=-1$ and $\boldsymbol{e}_1\boldsymbol{e}_2=-\boldsymbol{e}_2\boldsymbol{e}_1$. Then the conjugate $\bar{x}$ of an element $x=x_0\cdot 1+x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2+x_3\boldsymbol{e}_1\boldsymbol{e}_2$ of $\boldsymbol{Q}$ is given by $\bar{x}=x_0\cdot \mathbf{1}-x_1\boldsymbol{e}_1-x_2\boldsymbol{e}_2-x_3\boldsymbol{e}_1\boldsymbol{e}_2$. We define the Cayley algebra (the octanion algebra) $\mathfrak{C}=\boldsymbol{Q}+\boldsymbol{Q}\boldsymbol{e}$ by $(q+r\boldsymbol{e})\cdot(s+t\boldsymbol{e})=(qs-\bar{t}r)+(tq+r\bar{s})\boldsymbol{e}$ for $q,r,s,t\in\boldsymbol{Q}$. Then the conjugate $\bar{y}$ of an element $y=y_1+y_2\boldsymbol{e}$ of $\mathfrak{C}$ is given by $\bar{y}=\bar{y}_1-y_2e$ for $y_1, y_2\in \boldsymbol{Q}$. Put $A_1=\boldsymbol{R}\otimes_{\boldsymbol{R}}\boldsymbol{C}=\boldsymbol{C}\cdot\mathbf{1}$, $A_2=\boldsymbol{C}\otimes_{\boldsymbol{R}}\boldsymbol{C}=\boldsymbol{C}\cdot\mathbf{1}+\boldsymbol{C}\cdot\boldsymbol{e}_1$, $A_4=\boldsymbol{H}\otimes_{\boldsymbol{R}}\boldsymbol{C}=\boldsymbol{Q}$ and $A_8=\mathfrak{C}_{\boldsymbol{R}}\otimes_{\boldsymbol{R}}\boldsymbol{C}=\mathfrak{C}$. Let $\boldsymbol{V}_l$ be the totality of 3×3 hermitian matrices over A_l $(l=1,2,4,8)$ and let $\boldsymbol{G}_l$ be the group $\boldsymbol{SL}(3,A_l)$ $(l=1,2,4)$ and $\boldsymbol{E}_6$ $(l=8)$. Then the group $\boldsymbol{G}_l$ acts on $\boldsymbol{V}_l$ by $\rho_l(g)X=gX{}^t\bar{g}$ for $g\in\boldsymbol{G}$, $X\in\boldsymbol{V}_l$ $(l=1,2,4)$ and $\rho_8=\boldsymbol{\Lambda}_1$. Moreover, for $n\geq 1$, the group $\boldsymbol{G}_l\times \boldsymbol{GL}(n)$ has the action $\rho_l\otimes\boldsymbol{\Lambda}_1$ on $\boldsymbol{V}=\boldsymbol{V}_l\otimes\boldsymbol{V}(n)\cong\boldsymbol{V}_l\oplus\cdots\oplus\boldsymbol{V}_l$ (n-copies) by $X\mapsto(\rho_l(g_1)X_1,\cdots,\rho_l(g_1)X_n)g_2$, for $X=(X_1,\cdots,X_n)\in\boldsymbol{V}$ and $g=(g_1,g_2)\in\boldsymbol{G}_l\times\boldsymbol{GL}(n)$. This triplet $\boldsymbol{P}_{l,n}=(\boldsymbol{G}_l\times\boldsymbol{GL}(n),\ \rho_l\otimes\boldsymbol{\Lambda}_1,\ \boldsymbol{V}_l\otimes\boldsymbol{V}(n))$ is a regular irreducible prehomogeneous vector space for $n=1,2$ and $l=1,2,4,8$. In this article, we give the classification of their orbit spaces, the holonomy diagrams and the b-functions of their relative invariants.

In the case of $l=1$, this work was first done by Prof. M. Sato. In the case of $l=2$, this work was first done in the summar seminor for the study of the prehomogeneous vector spaces in 1974 by the participants including the authors, and reported by J. Sekiguchi in [4].

§1. Any relative invariant $f(X)$ of $\boldsymbol{P}_{l,n}$ $(n=1,2)$ is written as $f(X)=cf_{l,n}(X)^m$ $(c\in\boldsymbol{C}, m\in\boldsymbol{Z})$ with some irreducible polynomial $f_{l,n}(X)$. For an element X of $\boldsymbol{V}_l$, we can define the determinant $\det X$ (see [1]). Then we have $f_{l,1}(X)=\det X$ for $X\in\boldsymbol{V}_l$. For $n=2$, $f_{l,2}(X)$ is given by the discriminant $(z_1^2z_2^2+18z_0z_1z_2z_3-4z_0z_2^3-4z_1^3z_3-27z_0^2z_3^2)$ of the binary cubic form $\det(uX_1+vX_2)=\sum_{i=0}^3 z_iu^{3-i}v^i$ for $X=(X_1,X_2)\in\boldsymbol{V}_l\oplus\boldsymbol{V}_l$. We have $\deg f_{l,1}=3$ and $\deg f_{l,2}=12$.

§2. Put $\varphi(x)=\begin{pmatrix}x_0+\sqrt{-1}x_1, & -x_2-\sqrt{-1}x_3\\ x_2-\sqrt{-1}x_3, & x_0-\sqrt{-1}x_1\end{pmatrix}$ for $x=x_0\cdot 1+x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2+x_3\boldsymbol{e}_1\boldsymbol{e}_2\in\boldsymbol{Q}$. This gives an isomorphism $\varphi: A_4\rightrightarrows\boldsymbol{M}_2(\boldsymbol{C})$ which induces $A_2\rightrightarrows\left\{\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}; x,y\in\boldsymbol{C}\right\}$ and $A_1\rightrightarrows\left\{\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}; x\in\boldsymbol{C}\right\}$. We define the isomor-

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phism $\Phi: M(3, A_4)\to M(6, C)$ by $(x_{ij})\mapsto(\varphi(x_{ij}))$, which induces $SL(3, A_4) \cong SL(6, C)$. Put $J_1=\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $J=\begin{pmatrix} J_1 & & \\ & J_1 & \\ & & J_1 \end{pmatrix}$. Let $V(15)$ be the totality of 6×6 skew-symmetric matrices over C. Then, by $X\mapsto\Phi(X)J$, we have $V_4\cong V(15)$ and ρ_4 induces the action Λ_2 of $SL(6, C)$ on $V(15)$, i.e., $X\mapsto AX{}^tA$ for $A\in SL(6, C)$ and $X\in V(15)$. This implies that $P_{4,n}=(SL(6)\times GL(n), \Lambda_2\otimes\Lambda_1, V(15)\otimes V(n))$ $(n=1, 2)$. Now Φ induces $SL(3, A_2)\cong\{(B, C)\in GL(3, C)\times GL(3, C)\,;\ \det B\cdot\det C=1\}$ by $(a_{ij}) \mapsto((b_{ij}), (c_{ij}))$ for $\varphi(a_{ij})=\begin{pmatrix} b_{ij} & 0 \\ 0 & c_{ij}\end{pmatrix}$. By $(x_{ij})\mapsto(\varphi(x_{ij}))=\left(\begin{pmatrix} y_{ij} & 0 \\ 0 & z_{ij}\end{pmatrix}\right)\mapsto(y_{ij})$, we have $V_2\cong M(3, C)$, and ρ_2 induces the action $\Lambda_1\otimes\Lambda_1$ of $SL(3)\times SL(3)$ by $(y_{ij})\mapsto(b_{ij})(y_{ij}){}^t(c_{ij})$. This implies that $P_{2,n}=(SL(3)\times SL(3)\times GL(n), \Lambda_1\otimes\Lambda_1\otimes\Lambda_1, V(3)\otimes V(3)\otimes V(n))$ $(n=1, 2)$. Clearly, V_1 is the totality of 3×3 symmetric matrices and $\rho_1=2\Lambda_1$, i.e., $P_{1,n}=(SL(3)\times GL(n), 2\Lambda_1\otimes\Lambda_1, V(6)\otimes V(n))$ $(n=1, 2)$. The vector space V_8 becomes the exceptional simple Jordan algebra $\mathcal{J}$ by $X\cdot Y=(1/2)(XY+YX)$ for $X, Y\in V_8$.

§3. For $n=1$, it is simple and well-known (see [3]). Put $S_{l,k} =\rho_l(G_l)X_k$ with $X_k=\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}\in V_l$ $(l=1, 2, 4, 8\,;\ 0\leq k\leq 3)$. Then the $P.V.$ $P_{l,1}$ has four orbits $S_{l,k}$ $(0\leq k\leq 3)$ and their holonomy diagram is given in Fig. 1. Therefore the b-function $b(s)$ is given by $b(s) =(s+1)(s+1+(l/2))(s+1+l)$.

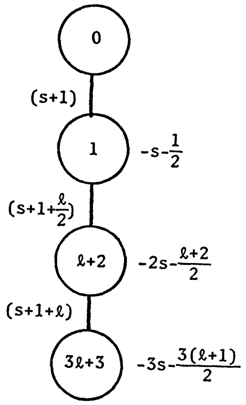

Fig. 1. Holonomy diagram of $P_{l,1}$.

§4. Our main purpose is to investigate the case for $n=2$. The classification of orbit spaces are given in Table I. The representative points in Table I are the points in $M(3, C)\oplus M(3, C)(\cong V_2\oplus V_2)$. The Zariski-closure of the conormal bundle of the orbit III_3 is not a good

Table I

	representative points (X_1, X_2)	orders	codimensions of orbits
I)	$\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\begin{pmatrix} & & \\ & 1 & \\ & & -1 \end{pmatrix}$	0	0
II$_1$)	$\begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \end{pmatrix}$	$-s-\frac{1}{2}$	1
II$_2$)	$\begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}\begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix}$	$-2s-\frac{l+1}{2}$	$l+1$
III$_1$)	$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}$	$-3s-\frac{3}{2}$	2
III$_2$)	$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}$	$-6s-\frac{2l+4}{2}$	$l+2$
III$_3$)	$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} & & \\ & 0 & \\ & & \end{pmatrix}$		$3l+2$
IV$_0$)	$\begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}$	$-4s-\frac{4}{2}$	4
IV$_1$)*	$\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}\begin{pmatrix} 1 & & \\ & & \\ & 1 & \end{pmatrix}$	$-4s-\frac{4}{2}$	4
IV$'_1$)*	$\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}\begin{pmatrix} 1 & & \\ & & 1 \\ & & \end{pmatrix}$	$-4s-\frac{4}{2}$	4
IV$_2$)*	$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & 1 \\ & 1 & \\ & & \end{pmatrix}$	$-6s-\frac{l+5}{2}$	$l+3$
IV$'_2$)*	$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & 1 & \\ 1 & & \end{pmatrix}$	$-6s-\frac{l+5}{2}$	$l+3$
V$^0_1$)	$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix}$	$-8s-\frac{2l+6}{2}$	$2l+2$
V$^1_1$)*	$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}$	$-8s-\frac{2l+6}{2}$	$\frac{5l}{2}+5$
V$^{1'}_1$)*	$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & & \\ 1 & & \\ & & \end{pmatrix}$	$-8s-\frac{2l+6}{2}$	$\frac{5l}{2}+5$
V$_2$)	$\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}$	$-9s-\frac{3l+6}{2}$	$2l+3$

V$_3$)	$\begin{pmatrix}1&&\\&1&\\&&\end{pmatrix}\begin{pmatrix}&&\\&0&\\&&\end{pmatrix}$	$-10s-\frac{5l+5}{2}$	$3l+3$
VI)	$\begin{pmatrix}1&&\\&&\\&&\end{pmatrix}\begin{pmatrix}&&\\&0&\\&&\end{pmatrix}$	$-11s-\frac{5l+6}{2}$	$4l+4$
VII)	$\begin{pmatrix}&&\\&0&\\&&\end{pmatrix}\begin{pmatrix}&&\\&0&\\&&\end{pmatrix}$	$-12s-\frac{6l+6}{2}$	$6l+6$

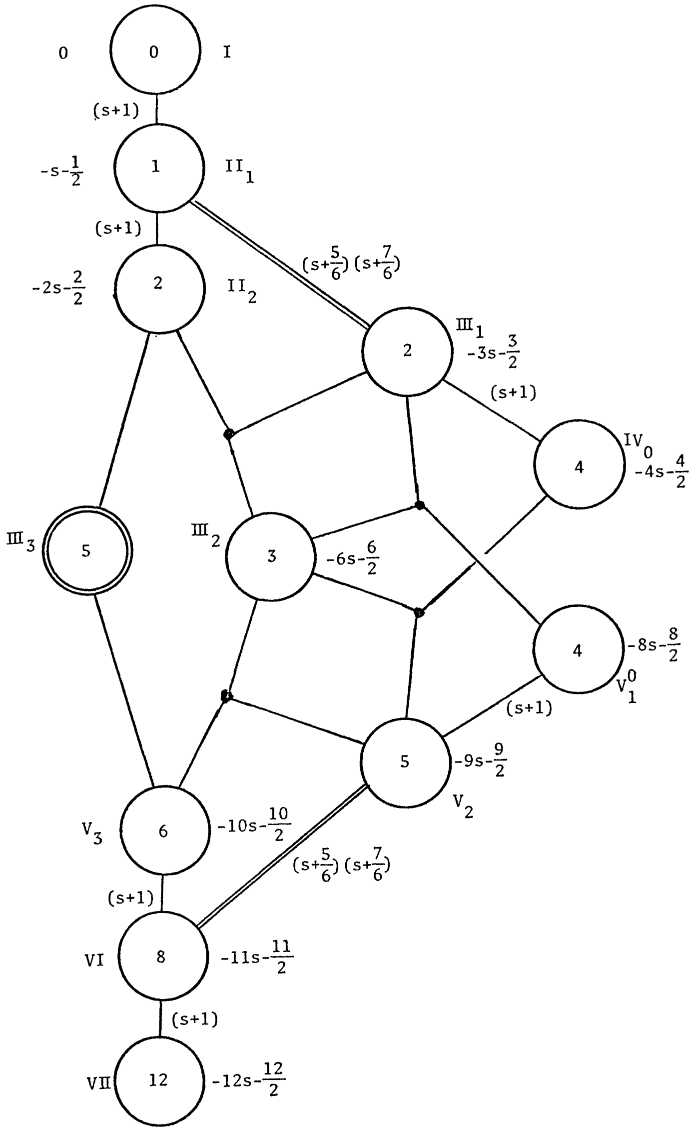

Fig. 2. Holonomy diagram of $P_{1,2}$.

Lagrangian subvariety (see [2]). The orbits with $*$ do not exist in the case for $l=1$. The orbits IV_1 (resp. IV_2, V_1^1) and IV_1' (resp. IV_2', $V_1^{1'}$) are different in the case for $l=2$, but they coincide in the case for $l=4,8$. Remarking these facts, one should see the holonomy diagram of $P_{l,2}$ for $l=2,4,8$ in Fig. 3.

The holonomy diagram of $P_{1,2}$ is given in Fig. 2. From these

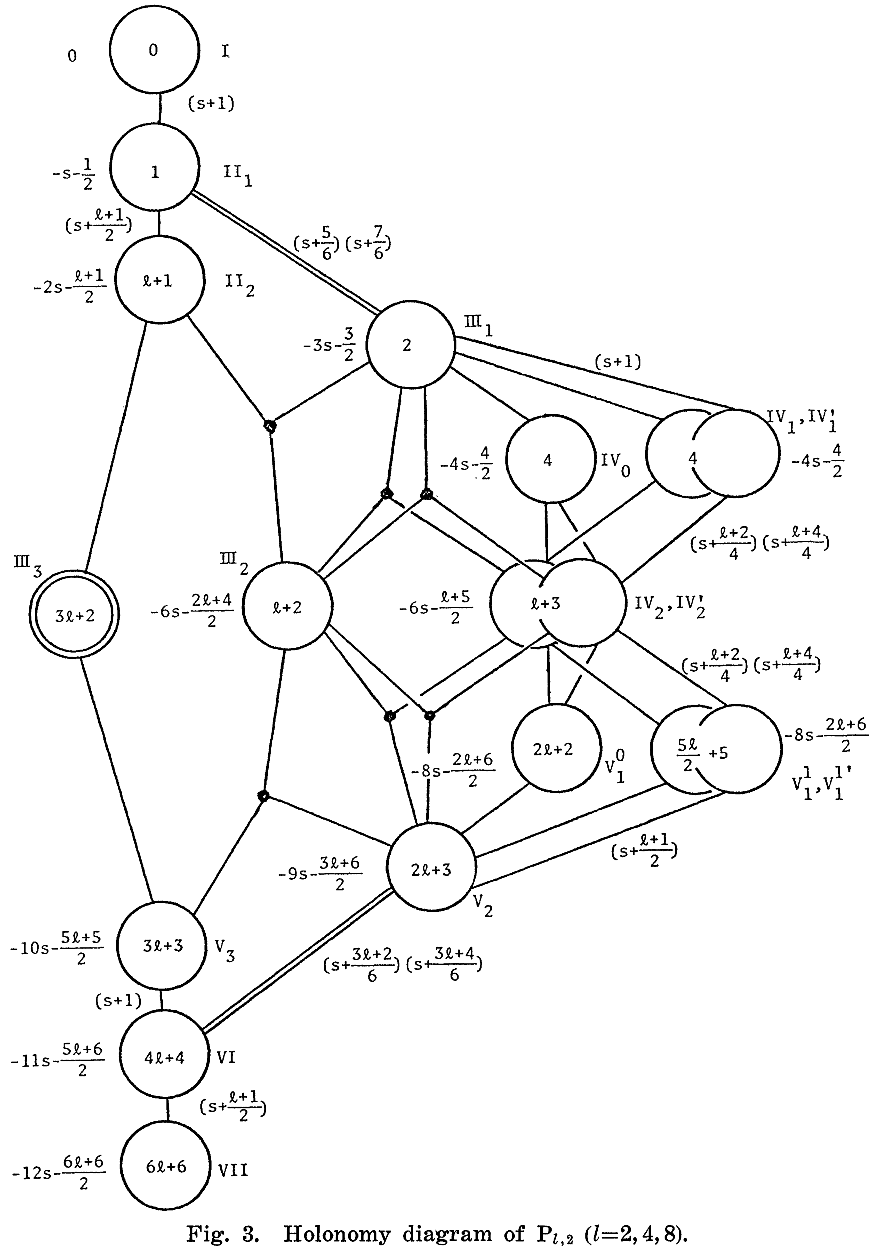

Fig. 3. Holonomy diagram of $P_{l,2}$ ($l=2,4,8$).

diagrams, we obtain the b-function

$$b(s)=(s+1)^2\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{l+2}{4}\right)^2\left(s+\frac{l+4}{4}\right)^2$$
$$\times\left(s+\frac{l+1}{2}\right)^2\left(s+\frac{3l+2}{6}\right)\left(s+\frac{3l+4}{6}\right),$$

i.e., $f_{l,2}(D_x)f_{l,2}^{s+1}(x)=b(s)f_{l,2}^{s}(x)$ $(s\in \boldsymbol{C})$ where

$$f_{l,2}(D_x)=f_{l,2}\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_N}\right)$$

with $N=\dim V_l\otimes V(2)=6l+6$, is a differential operator with constant coefficients (see [2]).

Remark. There exists a following similar series of regular irreducible prehomogeneous vector spaces (see [3] and [5]): (1) $(\boldsymbol{GL}(1)\times \boldsymbol{Sp}(3),\ \Lambda_1\otimes\Lambda_3,\ V(1)\otimes V(14))$, (2) $(\boldsymbol{GL}(6),\ \Lambda_3,\ V(20))$, (3) $(\boldsymbol{GL}(1)\times\boldsymbol{Spin}(12),\ \Lambda_1\otimes$half-spin rep., $V(1)\otimes V(32))$, (4) $(\boldsymbol{GL}(1)\times \boldsymbol{E}_7,\ \Lambda_1\otimes\Lambda_6,\ V(1)\otimes V(56))$.

References

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