87. *Infinitely Divisible Distributions and Ordinary Differential Equations*

By Takesi WATANABE
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1979)

1. Introduction. In the study of the limit distributions of multi-type Galton-Watson processes, S. Sugitani [2] has discovered that if a nonnegative function $\psi(t, \lambda)$, defined for $t \geqq 0$, $\lambda \geqq 0$, satisfies the ordinary differential equation having λ as a parameter

$$(1) \qquad \psi' = -B\psi^2 + \lambda p(t), \qquad \psi(0, \lambda) = m\lambda,$$

where $B > 0$, $m \geqq 0$ and $p(t)$ is a polynomial with positive coefficients, then there exists for each $t > 0$ an infinitely divisible distribution ν_t on $[0, \infty)$ such that

$$(2) \qquad \exp\left\{-\int_0^t \psi(s, \lambda)ds\right\} = \int_0^\infty e^{-\lambda x}\nu_t(dx).$$

Further information on ν_t is given in [3].

In this note we will prove a stronger result that $\exp\{-\psi(t, \lambda)\}$ is the Laplace transform of some infinitely divisible distribution μ_t on $[0, \infty)$. Our proof is quite elementary and can be applied to more general equations.

2. A heuristic argument. Given $f(x)$, $g(t, \lambda)$ and $h(\lambda)$ defined for $x \in (-\infty, \infty)$, $t \in [0, T]$, $\lambda \in [0, \infty)$, consider the following ordinary differential equation having λ as a parameter;

$$(3) \qquad \psi' = f(\psi) + g(t, \lambda), \qquad \psi(0, \lambda) = h(\lambda).$$

For the moment, we assume that equation (3) has a unique solution $\psi(t, \lambda)$ in $[0, T] \times [0, \infty)$. Here and after we will write ψ' for $D_t\psi(t, \lambda)$, $f^{(n)}$ for $D_x^n f$, $g_{n\lambda}(t, \lambda)$ for $D_\lambda^n g(t, \lambda)$ and so on. We now seek a suitable condition in order that *$\psi_\lambda(t, \lambda)$ is completely monotonic in $\lambda \in (0, \infty)$ for each $t \geqq 0$*. The essential part of our condition is that *$-f^{(2)}(\cdot)$, $g_\lambda(t, \cdot)$ for each $t \in [0, T]$ and $h_\lambda(\cdot)$ are completely monotonic in $(0, \infty)$*. To show the above assertion, differentiating (3) with respect to λ, we have

$$\psi_\lambda' = f^{(1)}(\psi)\psi_\lambda + g_\lambda(t, \lambda), \qquad \psi_\lambda(0, \lambda) = h_\lambda(\lambda).$$

Since $g_\lambda(t, \lambda) \geqq 0$ and $h_\lambda(\lambda) \geqq 0$, it follows that $\psi_\lambda(t, \lambda) \geqq 0$. Similarly, n-times differentiation of (3) leads us to

$$\begin{aligned}\psi_{n\lambda}' = f^{(1)}(\psi)\psi_{n\lambda} &+ f^{(2)}(\psi) \sum_{\substack{k_1+k_2=n \\ 1 \leqq k_1 \leqq k_2}} c_{k_1,k_2}\, \psi_{k_1\lambda}\psi_{k_2\lambda} \\ &+ \cdots + f^{(j)}(\psi) \sum_{\substack{k_1+\cdots+k_j=n \\ 1 \leqq k_1 \leqq \cdots \leqq k_j}} c_{k_1,\dots,k_j}\, \psi_{k_1\lambda} \cdot \cdots \cdot \psi_{k_j\lambda}\end{aligned}$$

$$+\cdots+f^{(n)}(\psi)\psi_\lambda^n+g_{n\lambda}(t,\lambda)$$
$$\psi_{n\lambda}(0,\lambda)=h_{n\lambda}(\lambda),$$

where each $c_{k_1,\dots,k_j}$ is a positive integer. By the induction hypothesis and our assumption for f, g, h, it is not difficult to see that $\psi_{n\lambda}\geqq 0$ or $\leqq 0$ according as n is odd or even. This formal argument will be justified under the condition (A) for f, g, and h given in Theorem 1.

Next suppose that $\psi(t,0)=0$. Then $h(0)=0$ and $f(0)+g(t,0)=0$. Therefore one can assume that

$$f(0)=0,\quad g(t,0)\equiv 0,\quad h(0)=0. \tag{4}$$

Under (4) and the previous assumption, we will show that *equation* (3) *has the unique nonnegative solution* $\psi(t,\lambda)$ *in* $(t,\lambda)\in[0,T]\times[0,\infty)$. But since $g(t,\lambda)\geqq 0$, $h(\lambda)\geqq 0$ by (4), $g_\lambda\geqq 0$ and $h_\lambda\geqq 0$, and since f is concave in $(0,\infty)$ by $-f^{(2)}\geqq 0$, it will be enough to prove (the first part of) the following

Lemma. *Let f be a differentiable concave function on $[0,\infty)$ with $f(0)=0$. Then for every nonnegative continuous function $g(t)$ on $[0,T]$ and every $\alpha\geqq 0$, the equation*

$$\psi'=f(\psi)+g(t),\qquad \psi(0)=\alpha \tag{5}$$

has the unique nonnegative solution $\psi(t)$ in $[0,T]$. In particular, if $\alpha>0$ and $g(t)>0$ except for $t=0$, then $\psi(t)>0$ in $[0,T]$.

Consider the following equation instead of (5),

$$\psi'=f(\psi^+)+g(t),\qquad \psi(0)=\alpha, \tag{6}$$

where $\psi^+=\psi\vee 0$. Suppose that the solution $\psi(t)$ of (6) has a point t_0 such that $\psi(t_0)<0$. Set $t_1=\sup\{t<t_0;\psi(t)=0\}$. By the mean value theorem, there exists $t_2\in(t_1,t_0)$ such that $\psi(t_2)<0$, $\psi'(t_2)<0$. This leads us to the contradiction that $\psi'(t_2)=f(\psi^+(t_2))+g(t_2)\geqq 0$. Therefore the solution of (6) should be nonnegative, so that $\psi^+=\psi$. Since f is concave, it follows that

$$0\leqq\psi(t)\leqq|f'(0)|\int_0^t\psi(s)ds+C\qquad\text{for } t\in[0,T],$$

where $C=\alpha+\int_0^T g(s)ds$. By Gronwall's inequality we get $\psi(t)\leqq C\exp(|f'(0)|t)$, which completes the proof of the first half of the lemma. For the latter half, suppose that $t_0=\inf\{t;\psi(t)=0\}<\infty$ ($t_0=\infty$ if $\{\ \}$ is empty). Since $t_0>0$ by $\alpha>0$, it follows that $\psi'(t_0)\leqq 0$, which contradicts to that $\psi'(t_0)=f(\psi(t_0))+g(t_0)=g(t_0)>0$.

3. We now have

Theorem 1. *Let $f(x)$, $g(t,\lambda)$ and $h(\lambda)$ be defined for $x\in[0,\infty)$, $t\in[0,T]$, $\lambda\in[0,\infty)^d=\{\lambda=(\lambda_1,\cdots,\lambda_d);\lambda_i\geqq 0\}$ and satisfy the following assumptions.*

(A.1) *$f(0)=0$, f is $C^1[0,\infty)\cap C^\infty(0,\infty)$ and $-f^{(2)}$ is CM (=completely monotonic) in $(0,\infty)$.*

(A.2) *$g(t,0)=0$, $g(t,\lambda)$ is $C([0,T]\times[0,\infty)^d)$,* C^∞ in $\lambda\in(0,\infty)^d$

$=\{\lambda=(\lambda_1, \cdots, \lambda_d); \lambda_i>0\}$ *for each* t *and all the partial derivatives* $g_{n_1\lambda_1,\cdots,n_d\lambda_d}(t, \lambda)=D_{\lambda_1}^{n_1}\cdots D_{\lambda_d}^{n_d} g(t, \lambda)$ *are* $C([0, T]\times(0, \infty)^d)$. *For each* t, *all the first partial derivatives* $g_{\lambda_i}(t, \lambda)$, $i=1, \cdots, d$, *are CM in* $\lambda\in(0, \infty)^d$.

(A.3) $h(0)=0$, $h(\lambda)$ *is* $C([0, \infty)^d)\cap C^\infty((0, \infty)^d)$ *and all the first partial derivatives* h_{λ_i} *are CM in* $\lambda\in(0, \infty)^d$.

(A.4) *At least one of the following conditions is fulfilled*: (a) $f^{(n)}(0+)$ *is finite for all* n, *or* (b) $h(\lambda)\not\equiv 0$ *and* $g(t, \lambda)\not\equiv 0$ *for each* $t>0$.

Under these assumptions we have the following.

(i) *For each* $\lambda\in[0, \infty)^d$, *equation* (3) *has the unique nonnegative solution* $\psi(t, \lambda)$ *in* $t\in[0, T]$. *For each* t, $\psi(t, \lambda)$ *is* $C^\infty((0, \infty)^d)$. $\psi(t, \lambda)$ *and all the partial derivatives* $\psi_{n_1\lambda_1,\cdots,n_d\lambda_d}(t, \lambda)$ *are* $C([0, T]\times(0, \infty)^d)$. $\psi(t, 0)\equiv 0$ *and all the first partial derivatives* $\psi_{\lambda_i}(t, \lambda)$ *are CM in* $\lambda\in(0, \infty)^d$ *for each* t.

(ii) $\psi(t, \lambda)$ *can be represented uniquely in the form of*

$$\psi(t, \lambda)=\langle c_t, \lambda\rangle+\int_{[0,\infty)^d}(1-e^{-\langle\lambda, y\rangle})n_t(dy) \tag{7}$$

by some $c_t\in[0, \infty)^d$ *and some measure* n_t *satisfying* $n_t(\{0\})=0$ *and* $\sum_{i=1}^{d}\int_{[0,\infty)^d}(y_i\wedge 1)n_t(dy)<\infty$.

(iii) *For each* $t\in[0, T]$, $\exp\{-\psi(t, \lambda)\}$ *is CM in* $\lambda\in(0, \infty)^d$ *and so it is the Laplace transform of an infinitely divisible distribution* μ_t *supported in* $[0, \infty)^d$;

$$\exp\{-\psi(t, \lambda)\}=\int_{[0,\infty)^d} e^{-\langle\lambda, x\rangle}\mu_t(dx). \tag{8}$$

Corollary. *For each* $t\in[0, T]$, *there exists an infinitely divisible distribution* ν_t *such that*

$$\exp\left\{-\int_0^t \psi(s, \lambda)ds\right\}=\int_{[0,\infty)^d} e^{-\langle\lambda, x\rangle}\nu_t(dx). \tag{9}$$

Consider the case $d=1$, since the extension to the multidimensional case is trivial. The first part of (i) follows from the preceding lemma. The rest of (i) will follow if the argument of the previous section is justified under assumption (A). To see this, it is enough to show that, for every n, $f^{(n)}(\psi(t, \lambda))$ is continuous in $[0, T]\times(0, \infty)$. This is obvious in the case (a) of (A.4). Condition (b) of (A.4) implies that $h(\lambda)>0$ and $g(t, \lambda)>0$ for every $t\in(0, T]$ and $\lambda\in(0, \infty)$ by (A.2) and (A.3). Therefore, by the latter half of the preceding lemma, $\psi(t, \lambda)>0$ for every $(t, \lambda)\in[0, T]\times(0, \infty)$ and hence $f^{(n)}(\psi(t, \lambda))$ is continuous.

Assertions (ii) and (iii) follow from the general theory on completely monotonic mapping (see [1]).

4. Further generalization. The result of the preceding section can be extended to much more general equation without difficulty.

Consider the equation

(10) $$\psi'=F(\psi, t, \lambda), \qquad \psi(0, \lambda)=h(\lambda).$$

Theorem 2. *Let $F(x, t, \lambda)$ be defined for $(x, t, \lambda) \in [0, \infty)\times[0, T]\times[0, \infty)^d$ and satisfy the following assumptions.*

(B.1) *$F(0, t, 0)=0$. $F(x, t, \lambda)$ is C in (x, t, λ), C^∞ in $(x, \lambda) \in (0, \infty)^{d+1}$ for each t and all the partial derivatives*

$$F^{(n)}_{n_1\lambda_1, \cdots, n_d\lambda_d}(x, t, \lambda)=D^n_x D^{n_1}_{\lambda_1}\cdots D^{n_d}_{\lambda_d}F(x, t, \lambda)$$

are C in $(0, \infty)\times[0, T]\times(0, \infty)^d$. For each t, all the first partial derivatives $F_{\lambda_i}(x, t, \lambda)$ is CM in $(x, \lambda) \in (0, \infty)^{d+1}$. For each (t, λ), $F(x, t, \lambda)$ is $C^1[0, \infty)$ and $-F^{(2)}(x, t, \lambda)$ is CM in $x \in (0, \infty)$.

(B.2)=(A.3) *of Theorem* 1.

(B.3) *At least one of the following conditions is fulfiled*: (a) *$F^{(n)}(0+, t, \lambda)$ is finite, or* (b) *$h(\lambda)\not\equiv 0$ and $F(0, t, \lambda)\not\equiv 0$ for each $t>0$.*

Then the same conclusions as in Theorem 1 *are valid for the solution $\psi(t, \lambda)$ of* (10).

References

[1] S. Bochner: Harmonic Analysis and the Theory of Probability. Univ. of California Press, Berkeley (1955).

[2] S. Sugitani: On the limit distributions of decomposable Galton-Watson processes. Proc. Japan Acad., **55**A, 334–336 (1979).

[3] ——: On the smoothness of the infinitely divisible distributions corresponding to some ordinary differential equations. Ibid., **55**A, 371–374 (1979).