# 64. *The Basis Problem for Modular Forms on $\Gamma_0(N)$*[†]

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§0. **Introduction.** Let $\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(Z) \mid c\equiv 0(N)\right\}$ and denote by $S_k(N, \psi)$ the space of cusp forms of weight $k \geqq 2$ and character $\psi$ on $\Gamma_0(N)$. M. Eichler ([5, p. 77]) formulated the "Basis Problem", roughly speaking to "construct explicitly" a basis of $S_k(N, \psi)$, as a generalization of a conjecture of Hecke ([6, Satz 53]) and presented a solution in the case $N$ is square free and $\psi=1$ ([3], [4], [5]). The purpose of this announcement is to sketch a "solution" for all weights $k \geqq 2$, all levels $N$, and all characters $\psi$ mod $N$.

Let $S_k^0(N, \psi)$ denote the subspace of $S_k(N, \psi)$ generated by newforms. As it is easy to obtain a basis of $S_k(N, \psi)$ if one knows a basis of $S_k^0(m, \psi)$ for $m \mid N$, we restrict our attention to $S_k^0(N, \psi)$. Eichler's result has been generalized ([10], [14]) to yield: If $N$ is not a square, $S_k^0(N, 1)$ is spanned by certain explicit theta series attached to quaternary quadratic forms associated to orders in $(p, \infty)$-quaternion algebras over $\boldsymbol{Q}$ (i.e. ramified at $p$ and $\infty$), for various prime divisors $p$ of $N$. If $N$ is a square, such a result cannot hold in general. Using calculations of Parry [12], A. O. L. Atkin was able to discover in the case $S_2^0(13^2, 1)$ which newforms are not obtained from theta series and his questions and ideas about this to one of the present authors led to the "solution" for the case $S_k^0(p^2M, 1)$, $p$ an odd prime, $p \nmid M$ ([15]).

Our general solution which includes all the above as special cases goes as follows. Call $S_k(N, \varphi)$ a *primitive neben space* if cond $(\varphi)=N$. An eigenform for the Hecke operators $T(n)$, $(n, N)=1$ in such a space will be called a *primitive nebenform*. Our main result shows how to explicitly decompose $S_k^0(N, \psi)$ as a direct sum of character twists of primitive neben spaces and twists of spaces spanned by certain "theta series" associated to $(p, \infty)$-quaternion algebras. That this is a reasonable solution to the basis problem follows from the result: For a newform $f$ in $S_k^0(N, \psi)$ corresponding to the representation $\pi=\otimes\pi_\ell$ of the

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adele group $GL_2$, the following three conditions are equivalent: (1) $f$ is not representable by theta series of any $(p, \infty)$-quaternion algebra; (2) $\pi_\ell$ belongs to the principal series for all $\ell<\infty$; and (3) $f$ is the twist of some primitive nebenform. This result can be derived from Theorem 16.1 of Jacquet-Langlands [11]. As in Eichler's original proof, most of our results are consequences of trace identities.

**§1. Character twists of newforms.** Let $f=f(\tau)=\sum a(n)x^n \in S_k(N, \psi)$, $x=e^{2\pi i\tau}$. If $\mu$ is a primitive (Dirichlet) character we denote by $f^\mu$ the twist of $f$ by $\mu$, $f^\mu=\sum \mu(n)a(n)x^n$. $f^\mu \in S_k(N', \psi\mu^2)$ for some $N'$. We let $S_k^0(N, \psi)^\mu=\{f^\mu \mid f \in S_k^0(N, \psi)\}$, etc. Throughout this note $p$ denotes a prime, $M$ any positive integer prime to $p$, $\omega$, $\chi$, and $\lambda$ characters mod a power of $p$, and $\varphi$ any character mod $M$. We define $e(\omega)$, etc. by cond $(\omega)=p^{e(\omega)}$ and $e(1)=0$.

**Theorem 1.1.** *If $e(\omega)>r/2$, then*

$$S_k^0(p^rM, \omega\varphi)=\bigoplus_\chi S_k^0(p^{e(\omega)}M, \omega\chi^2\varphi)^{\bar\chi}$$

*where the sum is over all primitive characters $\chi$ mod $p^{r-e(\omega)}$.*

**Theorem 1.2.** *If $e(\omega)<r/2$ and $e(\chi)+e(\omega)<r$, then*

$$S_k^0(p^rM, \omega\chi^2\varphi)=S_k^0(p^rM, \omega\varphi)^\chi.$$

**Theorem 1.3.** *If $e(\chi)<r$, then*

$$S_k^0(p^rM, \chi\varphi)=S_k^0(p^rM, \bar\chi\varphi)^\chi.$$

**Theorem 1.4** ([17] if $\omega\varphi=1$). *Let $p=2$ and $m\geqq 2$. If $e(\omega)<m$ and $\lambda$ is any primitive character* mod $2^m$, *then*

$$S_k^0(2^{2m}M, \omega\varphi)=\bigoplus_{f=e(\omega\lambda^2)}^{2m-1} S_k^0(2^fM, \omega\lambda^2\varphi)^\lambda.$$

**Theorem 1.5.** *Let $f=\sum a(n)x^n$ be a newform in $S_k^0(N, \psi)$. Let $q$ be a prime dividing $N$. Then $a(q)=0$ if and only if $f=g^\mu$ for some newform $g$ and some character $\mu$ with $q \mid$ cond $(\mu)$.*

**Remarks.** The results in this section should be compared with those in Atkin-Li [1]. Also for brevity we have not stated results which hold for the cases $e(\omega)\leqq r/2$, $e(\chi)=r/2$ in Theorem 1.2, $e(\chi)=r$ in Theorem 1.3 and $e(\omega)=m$ in Theorem 1.4. The above theorems are proved by applying the trace formula of [9]. However with the exception of Theorem 1.4, they can also be derived by computing the conductors of the corresponding representations.

**§2. Theta series.** Let $B$ be a $(p, \infty)$-quaternion algebra over $\boldsymbol{Q}$, $R$ an order of $B$, and $M$ a positive integer prime to $p$. For a prime $\ell$, let $R_\ell=R\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_\ell$. We consider orders $R$ of $B$ such that (i) $R\cong\begin{pmatrix}\boldsymbol{Z}_\ell & \boldsymbol{Z}_\ell \\ M\boldsymbol{Z}_\ell & \boldsymbol{Z}_\ell\end{pmatrix}$ for all $\ell\neq p$ and (ii) $R_p$ contains a subring isomorphic to the ring of integers in some quadratic field extension $L$ of $\boldsymbol{Q}_p$. Such orders are determined up to local isomorphisms by $M$, $L$, and the "level" at $p$ which is a power of $p$, say $p^s$. In special cases such orders have been studied in [2], [8], [13], and [15]. If $\varphi$ is a character mod $M$ and $\omega$ a character mod a suitable power of $p(\leqq p^s)$, we can in a manner similar to [5, p. 110] and [7, p. 586] (also see [18, § 2]) associate to such orders Brandt Matrices with character, say $B_k(n)=B_k(n;R,\omega,\varphi)=B_k(n;L,p^s,\omega;M,\varphi)$ for $k\geqq 2$ and $n\geqq 0$. In theory (and often in practice, see [16]) the $B_k(n)$ can be explicitly computed.

**Theorem 2.1.** *The entries $\theta_{ij}(\tau)$ of the Brandt Matrix series*

$$(\theta_{ij}(\tau))=\sum_{n=0}^{\infty} B_k(n;L,p^s,\omega;M,\varphi)e^{2\pi i n\tau}$$

*are modular forms of weight $k$ and character $\omega\varphi$ on $\Gamma_0(N^2)$ where $N=p^sM$. If $k>2$ or $\varphi\neq 1$ or $\omega$ is odd, the $\theta_{ij}(\tau)$ are cusp forms.*

The $B_k(n)$ for $(n,pM)=1$ can be simultaneously diagonalized say by conjugation by a matrix $A$. Let $\sum AB_k(n)A^{-1}e^{2\pi i n\tau}=\begin{pmatrix}\theta_1 & & * \\ & \ddots & \\ * & & \theta_r\end{pmatrix}$. If $k>2$ or $\varphi\neq 1$ or $\omega$ is odd let $\Theta_k(R,\omega\varphi)=\Theta_k(L,p^s,\omega;M,\varphi)=\langle\theta_1\rangle\oplus\cdots\oplus\langle\theta_r\rangle$ where $\langle\theta_i\rangle$ denotes the 1 dim. complex vector space spanned by $\theta_i$. In the other cases some of the $\theta_i$ may be eisenstein series and we let $\Theta_k(R,\omega\varphi)$ be the direct sum of the $\langle\theta_i\rangle$ after deleting the eisenstein series. Let $H$ denote the Hecke algebra generated by all $T(n)$, $(n,pM)=1$. $\Theta_k(R,\omega\varphi)$ is (via the action of the Brandt Matrices) an $H$-module and the $\theta_i$ are eigenforms for all $T(n)$, $(n,pM)=1$. An equivalent non-constructive but more conceptical and representation theoretic definition of $\Theta_k(R,\omega\varphi)$ is obtained by considering functions on the idele group of $B$ as in § 2 of Shimizu [18]. As in [10, p. 19] and [7, p. 586] we can define the subspace $\Theta_k^0(R,\omega\varphi)$ of "newforms" in $\Theta_k(R,\omega\varphi)$.

For the solution of the basis problem it suffices to consider $\omega$ of low conductor. Hence let $\varepsilon=1$ or any odd character mod $p$ (4 if $p=2$). Also we write $A\cong B$ to mean $A$ and $B$ are isomorphic as $H$-modules.

**Theorem 2.2** ([4] if $s=0$, $M$ square free, and $\varepsilon\varphi=1$, [10] if $s=0$ and $\varepsilon\varphi=1$, [14] if $\varepsilon\varphi=1$). *If $s\geqq e(\varepsilon)$, then*

$$S_k^0(p^{2s+1}M,\varepsilon\varphi)\cong\Theta_k^0(L_0,p^{2s+1},\varepsilon;M,\varphi)$$

*where $L_0$ is the unramified quadratic extension of $\boldsymbol{Q}_p$.*

**Theorem 2.3.** *If $p$ is odd and $s\geqq 2$, then*

$$2S_k^0(p^{2s}M,\varepsilon\varphi)\cong\Theta_k^0(L_1,p^{2s},\varepsilon;M,\varphi)\oplus\bigoplus_{\lambda/\sim}2S_k^0(p^sM,\varepsilon\lambda^2\varphi)^\lambda$$

*where $L_1$ is either ramified quadratic extension of $\boldsymbol{Q}_p$ and the sum $\bigoplus_{\lambda/\sim}$ is over all primitive characters* mod $p^s$ *modulo the equivalence $\lambda\sim\overline{\varepsilon\lambda}$.*

**Theorem 2.4.** *Let $p=2$ and $s\geqq 4$ and fix a character $\lambda$ primitive* mod $2^s$. *Then* $4S_k^0(2^{2s-2}M,\varepsilon\lambda^2\varphi)^\lambda\oplus 2S_k^0(2^{2s-1}M,\varepsilon\lambda^2\varphi)^\lambda\cong\Theta_k^0(L_2,2^{2s},\varepsilon;M,\varphi)\oplus\Theta_k^0(L_6,2^{2s},\varepsilon;M,\varphi)$ *where $L_2=\boldsymbol{Q}_2(\sqrt{2})$, $L_6=\boldsymbol{Q}_2(\sqrt{6})$.*

**Remarks.** Results similar to those above hold for $S_k^0(p^2M,\varepsilon\varphi)$, $p$ odd (see [15]) and for $S_k^0(2^sM,\varepsilon\varphi)$, $s=2$ or $4$. Theorem 1.3 is valid when $e(\chi)=r$ if we replace $=$ by $\cong$ and this shows the $\cong$ in Theorem

2.3 is well defined.

**§3. The basis problem.** The idea is to use the theorems of §1 to reduce to the cases covered by the theorems of §2. If $\psi$ is a character mod $N$, let $\psi=\prod_{\ell|N}\psi_\ell$ where $\psi_\ell$ is a character mod $\ell^f$, $f=\mathrm{ord}_\ell(N)$. We call $\Theta_k^0(L, p^s, \varepsilon; M, \varphi)$ or a subspace of it a *p-theta space with character* $\varepsilon$.

**Theorem 3.1.** *Any space of newforms* $S_k^0(N, \psi)$ *with* $k\geqq 2$ *can be explicitly decomposed into a direct sum of twists of primitive neben spaces and twists of p-theta spaces with character* $\varepsilon$ *where p ranges over prime divisors of N and for each* $p, \varepsilon=1$ *or any odd character* mod $p$ (4 if $p=2$).

This follows from

**Proposition 3.2.** *Let* $p|N$. $S_k^0(N, \psi)$ *can be explicitly decomposed into a direct sum of twists of p-theta spaces with character* $\varepsilon$ *and twists of spaces* $S_k^0(N', \psi')$ *satisfying* $e(\psi'_p)=\mathrm{ord}_p(N')$, $\mathrm{ord}_\ell(N')=\mathrm{ord}_\ell(N)$ *for* $\ell\neq p$ *and* $\prod_{\ell\neq p}\psi'_\ell=\prod_{\ell\neq p}\psi_\ell$.

We indicate the proof in most cases. Let $N=p^rM$ and $\psi=\omega\varphi$ with $\omega$ mod $p^r$ and $\varphi$ mod $M$. If $e(\omega)>r/2$, apply Theorem 1.2. If $r=2s+1$ and $e(\omega)\leqq s$ if $p\neq 2$ or $e(\omega)<s$ if $p=2$, then $\omega=\varepsilon\chi^2$ for some $\chi$ with $e(\chi)\leqq s$ and $\varepsilon$ as above. Now apply Theorems 1.2 and 2.2. If $r=2s\geqq 4$, $p\neq 2$, and $e(\omega)<s$, then $\omega=\varepsilon\chi^2$ and apply Theorems 1.2 and 2.3. As a final example consider the case $p=2$, $r=2s\geqq 8$, and $e(\omega)\leqq s-2$. Then $\omega=\varepsilon\chi^2$ with $e(\chi)\leqq s-1$ and apply Theorems 1.2, 1.4 and 2.5.

**Corollary 3.3.** *If* $e(\psi_\ell)>(1/2)\,\mathrm{ord}_\ell(N)$ *for all* $\ell|N$, *then*

$$S_k^0(N, \psi)=\bigoplus_{\mu} S_k^0(\mathrm{cond}(\psi), \psi\mu^2)^{\bar{\mu}}$$

*where the sum is over all* $\mu$ *primitive* mod $(N/\mathrm{cond}(\psi))$.

In the case covered by Corollary 3.3 all newforms are twists of primitive nebenforms. In all other cases in general there will be newforms that arrise as the twists of "theta series" associated to definite quaternion algebras.

We should stress that the case of prime power level $N=p^s$ is the essential case and in this case ($M=\varphi=1$), the results of §§1 and 2 are complete in and of themselves. Generalizing to the case of non-prime power level is easy. In the general case the $p$-theta spaces appearing in our solution to the basis problem in §3 are not unique, e.g. $S_k^0(p, q)\cong\Theta_k^0(L_0, p, 1; q, 1)\cong\Theta_k^0(L'_0, q, 1; p, 1)$. A better treatment involves considering arbitrary quaternion algebras, not just those ramified at a single finite prime. However, these results are too complicated to describe here. In fact most of our results can be generalized to arbitrary quaternion algebras over totally real number fields.

Our results can be viewed as a refinement of Theorem 16.1 of Jacquet-Langlands [11] in the holomorphic case. In particular our results show how to construct from theta series attached to $(p, \infty)$-quaternion algebras all newforms corresponding to representations $\pi = \otimes \pi_\ell$ of $GL_2$ where $\pi_\ell$ is not in the principal series (i.e. is special or supercuspidal) for at least one $\ell < \infty$.

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