89. A Variational Problem Relating to the Theory of Optimal Economic Growth*)

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1. Introduction. The theory of optimal economic growth is one of the most attractive themes in the recent developments in mathematical economics. The basic problem is to find out an optimal path of economic growth (or capital accumulation) in the sense that it maximizes certain economic welfare over time under some technological constraint. Being stimulated by the ingenious idea of F. P. Ramsey [6], a lot of economists, including P. A. Samuelson and T. C. Koopmans, have been working on this field and various mathematical theories of optimal control such as the Pontrjagin's maximum principle have been successfully introduced to economic analysis.

Recently, Chichilnisky [2] tried to prove rigorously the existence of an optimal path of economic growth relying upon an effective use of the weighted Sobolev space. And Takekuma [7] also gave another interesting version of the existence proof. The purpose of the present paper is to add a further new insight to this existence problem, and the author is much indebted to Berkovitz [1] for the basic ideas embodied in the proof.

2. Problem. Let us begin with specifying some notations and their economic interpretations. First the following items are assumed to be given.

$[0, T]$ planning time horizon.

$u: \boldsymbol{R}^l_+ \to \boldsymbol{R}_+$ welfare function.

$f: \boldsymbol{R}^l_+ \to \boldsymbol{R}^l_+$ production function at time 0.

$\rho > 0$ the rate of technological progress.

$\delta > 0$ the discount rate of the welfare in the future.

$\lambda \in (0, 1)^l$ the vector of the depreciation rates of l capital goods.

Furthermore we have a couple of variable mappings to be optimized:

$k: [0, T] \to \boldsymbol{R}^l_+$ path of capital accumulation.

$s: [0, T] \to [0, 1]^l$ path of the vector whose components are saving rates of each goods.

For any vector $x \in \boldsymbol{R}^l$, we designate by M_x the diagonal matrix of the form

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$$M_x=\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_l \end{pmatrix}$$

where x_i $(1\leqq i\leqq l)$ is the i-th coordinate of x.

Then the problem of optimal economic growth can be formulated as follows:

(P) *Maximize*

$$J(k, s)=\int_0^T u[(I-M_{s(t)})f(k(t))e^{\rho t}]e^{-\delta t}dt \tag{1}$$

subject to

$$\dot{k}(t)=M_{s(t)}f(k(t))e^{\rho t}-M_\lambda k(t) \tag{2}$$

$$k(0)=\bar{k} \text{ (given vector)}. \tag{3}$$

(I is the identity matrix.)

If we define $w:[0, T]\times \boldsymbol{R}^l_+\times[0,1]^l\to\boldsymbol{R}_+$ and $g:[0, T]\times\boldsymbol{R}^l_+\times[0,1]^l\to\boldsymbol{R}^l$ by

$$w(t, k, s)=u[(I-M_s)f(k)e^{\rho t}]e^{-\delta t}$$

and

$$g(t, k, s)=M_sf(k)e^{\rho t}-M_\lambda k$$

respectively, then the problem (P) can be rewritten in the form:

(P′) *Maximize*

$$J(k, s)=\int_0^T w(t, k(t), s(t))dt \tag{1′}$$

subject to

$$\dot{k}(t)=g(t, k(t), s(t)) \tag{2′}$$

$$k(0)=\bar{k}. \tag{3′}$$

(Consult Mityagin [5] for the economic interpretation of the above problem!)

Throughout this paper, we shall assume the following conditions to be satisfied.

Assumption 1. u is continuous and concave.

Assumption 2. f is continuous.

Assumption 3. There exists $C>0$ such that

$$k_i\geqq C \quad \text{implies} \quad f_i(k)e^{\rho T}\leqq\lambda_i k_i$$

for any i $(=1, 2, \cdots, l)$, where k_i (resp. f_i) is the i-th coordinate of k (resp. f).

3. Boundedness of admissible paths. We denote by $\mathcal{S}$ the set of all the measurable mappings $s:[0, T]\to[0,1]^l$.

Definition. A pair (k, s) $W^{1,2}\times\mathcal{S}$ is said to be an *admissible pair* if it satisfies (2) and (3). And $k\in W^{1,2}$ is called an *admissible path* if there exists an $s\in\mathcal{S}$ such that (k, s) is an admissible pair. The set of all the admissible pairs is denoted by $\mathcal{A}$, and the set of all the admissible paths by $\mathcal{A}_k$.

Proposition 1. *$\mathcal{A}_k$ is bounded in $W^{1,2}$.*

Proof. Since

$$\dot{k}(t)+M_\lambda k(t)=M_{s(t)}f(k(t))e^{\rho t}\geqq 0,$$

we must have

$$\dot{k}_i(t)\geqq -\lambda_i k_i(t) \qquad \text{for all } i.$$

Hence

$$k_i(t)\geqq \bar{k}_i e^{-\lambda_i t}\geqq \bar{k}_i e^{-\lambda_i T} \qquad \text{for all } i. \tag{4}$$

On the other hand, since

$$f_i(k(t))e^{\rho t}\leqq \lambda_i k_i(t) \qquad \text{if } k_i(t)\geqq C.$$

by Assumption 3, we must have

$$\dot{k}_i(t)=s_i(t)f_i(k(t))e^{\rho t}-\lambda_i k_i(t)\leqq 0 \qquad \text{if } k_i(t)\geqq C. \tag{5}$$

Consequently

$$k_i(t)\leqq C \qquad \text{for all } i. \tag{6}$$

By (4) and (6),

$$\bar{k}_i e^{-\lambda_i T}\leqq k_i(t)\leqq C\,; \qquad \text{for all } t \text{ and } i, \tag{7}$$

that is, $\mathcal{A}_k$ is uniformly essentially bounded.

Furthermore, since (7) and Assumption 2 imply that

$$-\lambda_i C\leqq \dot{k}_i(t)\leqq \sup\{f_i(k(t))e^{\rho T}\mid k\in\mathcal{A}_k,\ t\in[0,T]\}<\infty \qquad \text{for all } i, \tag{8}$$

$\{\dot{k}\mid k\in\mathcal{A}_k\}$ is also uniformly essentially bounded. Thus we can conclude that $\mathcal{A}_k$ is bounded in $W^{1,2}$. Q.E.D.

Corollary 1. *$\mathcal{A}_k$ is weakly sequentially compact in $W^{1,2}$.*

Proof. Since $W^{1,2}$ is a Hilbert space, the boundedness of $\mathcal{A}_k$ implies that it is weakly sequentially compact. Q.E.D.

4. Existence theorem. **Proposition 2.** *$\gamma\equiv\sup_{(k,s)\in\mathcal{A}}J(k,s)$ is finite.*

Proof. As we have already proved in (7), $\mathcal{A}_k$ is uniformly essentially bounded. Hence, by Assumption 2,

$$\sup\{\|(I-M_{s(t)})f(k(t))e^{\rho t}\|\mid(k,s)\in\mathcal{A},\ t\in[0,T]\}$$

is finite. By the continuity of u (Assumption 1),

$$\sup\{w(t,k(t),s(t))\mid(k,s)\in\mathcal{A},\ t\in[0,T]\}$$

is also finite. Therefore γ must be finite. Q.E.D.

Let us define the correspondence (=multi-valued mapping) $\Omega:[0,T]\times\boldsymbol{R}^l_+\twoheadrightarrow\boldsymbol{R}^l\times\boldsymbol{R}_+$ by

$$\Omega(t,k)=\{(\xi,\eta)\in\boldsymbol{R}^l\times\boldsymbol{R}_+\mid\xi=g(t,k,s)\text{ and }0\leqq\eta\leqq w(t,k,s) \text{ for some } s\in[0,1]^l\}. \tag{9}$$

Thanks to our Assumptions 1 and 2, it is quite easy to prove that Ω is a compact-convex-valued continuous correspondence. Therefore the correspondence

$$k\twoheadrightarrow\Omega(\tilde{t},k)=\overline{\mathrm{co}}\ \Omega(\tilde{t},k) \tag{10}$$

is also a compact-convex-valued continuous correspondence for each fixed $\tilde{t}\in[0,T]$. If we denote

$$K(\tilde{t}\,;\tilde{k},\varepsilon)=\{(\tilde{t},k)\in[0,T]\times\boldsymbol{R}^l_+\mid\|k-\tilde{k}\|<\varepsilon\}$$

($(\tilde{t}, \tilde{k}) \in [0, T] \times \boldsymbol{R}_+^l$), then we obtain the following result as a consequence of the continuity of the correspondence (10).

Proposition 3. *For each* $(\tilde{t}, \tilde{k}) \in [0, T] \times \boldsymbol{R}_+^l$,

$$\Omega(\tilde{t}, \tilde{k}) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, \Omega(K(\tilde{t}\,; \tilde{k}, \varepsilon)).$$

Thus we have just finished up the preparation for the following crucial proposition.

Let $\{(k_n, s_n)\}$ be a sequence in $\mathcal{A}$ such that

$$\lim_n J(k_n, s_n) = \gamma. \tag{11}$$

Then, by Corollary 1, there exists a weakly convergent subsequence (no change in notations) of $\{k_n\}$; i.e.

$$k_n \longrightarrow k^* \qquad \text{weakly in } W^{1,2}. \tag{12}$$

Proposition 4. *There exists an integrable function* $\zeta : [0, T] \to \boldsymbol{R}$ *such that*

$$\int_0^T \zeta(t)dt \geqq \gamma \tag{13}$$

and

$$(\dot{k}^*(t), \zeta(t)) \in \Omega(t, k^*(t)) \qquad a.e. \tag{14}$$

Proof. (12) implies that $k_n \to k^*$ strongly in L^2. Hence we can assume, without loss of generality, that

$$k_n(t) \longrightarrow k^*(t) \qquad \text{a.e.} \tag{15}$$

On the other hand, (12) implies that

$$\dot{k}_n \longrightarrow \dot{k}^* \qquad \text{weakly in } L^2. \tag{16}$$

Therefore, by the well-known Mazur's theorem, we can find out, for each $j \in \boldsymbol{N}$, some finite elements

$$k_{n_j+1}, k_{n_j+2}, \cdots, k_{n_j+m(j)}$$

in $\{k_n\}$ and

$$\alpha_{ij} \geqq 0, \quad 1 \leqq i \leqq m(j), \quad \sum_{i=1}^{m(j)} \alpha_{ij} = 1$$

such that

$$\left\| \dot{k}^* - \sum_{i=1}^{m(j)} \alpha_{ij} \dot{k}_{n_j+i} \right\|_2 \leqq \frac{1}{j}, \qquad n_{j+1} > n_j + m(j). \tag{17}$$

We denote

$$\begin{aligned} \psi_j(t) &= \sum_{i=1}^{m(j)} \alpha_{ij} \dot{k}_{n_j+i}(t) \\ &= \sum_{i=1}^{m(j)} \alpha_{ij} g(t, k_{n_j+i}(t), s_{n_j+i}(t)). \end{aligned} \tag{18}$$

By (17), we can assume, without loss of generality, that

$$\psi_j(t) \longrightarrow \dot{k}^*(t) \qquad \text{a.e.} \tag{19}$$

Define a sequence of functions $\{\zeta_j : [0, T] \to \boldsymbol{R}\}$ by

$$\zeta_j(t) = \sum_{i=1}^{m(j)} \alpha_{ij} w(t, k_{n_j+i}(t), s_{n_j+i}(t)). \tag{20}$$

And if we define

$$\zeta(t) = \operatorname{limsup} \zeta_j(t), \tag{21}$$

then $\zeta(t)$ is bounded as proved in Proposition 2. Applying the Fatou's lemma, we must have the following inequality:

$$\limsup \int_0^T \zeta_j(t)dt \leqq \int_0^T \limsup \zeta_j(t)dt = \int_0^T \zeta(t)dt. \tag{22}$$

It is easy to show that

$$\limsup \int_0^T \zeta_j(t)dt = \gamma. \tag{23}$$

Combining (22) with (23), we get (13).

It remains to show (14). For each fixed t, we can assume that

$$\zeta_j(t) \longrightarrow \zeta(t). \tag{24}$$

Taking account of (15), we can find out some $n_0 \in \boldsymbol{N}$, for each $\varepsilon > 0$, such that

$$\|k_n(t) - k^*(t)\| < \varepsilon \qquad \text{for all } n \geqq n_0. \tag{25}$$

Therefore

$$(t, k_n(t)) \in K(t\,;\, k^*(t), \varepsilon) \qquad \text{for all } n \geqq n_0. \tag{26}$$

Consequently we have, for sufficiently large j,

$$(g(t, k_{n_{j+i}}(t), s_{n_{j+i}}(t)),\ w(t, k_{n_{j+i}}(t), s_{n_{j+i}}(t))) \in \Omega(K(t\,;\, k^*(t), \varepsilon)), \tag{27}$$

which implies

$$(\psi_j(t), \zeta_j(t)) \in \operatorname{co} \Omega(K(t\,;\, k^*(t), \varepsilon)). \tag{28}$$

Furthermore by (19) and (24),

$$(\dot{k}^*(t), \zeta(t)) \in \overline{\operatorname{co}}\, \Omega(K(t\,;\, k^*(t), \varepsilon)). \tag{29}$$

Since (29) holds for arbitrary $\varepsilon > 0$,

$$(\dot{k}^*(t), \zeta(t)) \in \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\, \Omega(K(t\,;\, k^*(t), \varepsilon)) = \Omega(t, k^*(t)). \tag{30}$$

The last equality comes from Proposition 3. This completes the proof. Q.E.D.

By Proposition 4, it has been verified that the value

$$\int_0^T \zeta(t)dt$$

can be attained under the path $k^*(t)$ if $s^*(t) \in [0, 1]^l$ is suitably chosen at each $t \in [0, T]$. Finally we shall prove that $s(t)$ can be chosen so as to be measurable. Although this point is almost obvious in our simple case, it may be suggestive, for the sake of further sophistications of the problem, to provide another proof based on the Filippov's implicit function theorem (cf. Maruyama [4, pp. 477–478]).

Define the mapping $\varphi: [0, T] \times [0, 1]^l \to \boldsymbol{R}^l \times \boldsymbol{R}_+$ and the correspondence $\Phi: [0, T] \longrightarrow\!\!\!\!\rightarrow \boldsymbol{R}^l \times \boldsymbol{R}_+$ by

$$\varphi(t, s) = (g(t, k^*(t), s),\ w(t, k^*(t), s)), \qquad \Phi(t) = \varphi(t, [0, 1]^l).$$

Furthermore if we define the correspondence $\Gamma: [0, T] \longrightarrow\!\!\!\!\rightarrow \boldsymbol{R}^l \times \boldsymbol{R}_+$ by

$$\Gamma(t) = \{(\dot{k}^*(t), \eta) \in \boldsymbol{R}^l \times \boldsymbol{R}_+ \mid \zeta(t) \leqq \eta\},$$

then, by Proposition 4, we must have

$$\Gamma(t) \cap \Phi(t) \neq \phi.$$

Since every condition required for the Filippov's theorem is trivially satisfied, there exists a measurable mapping $s^*: [0, T] \to [0, 1]^l$ such that

$$\varphi(t, s^*(t)) = (g(t, k^*(t), s^*(t)), w(t, k^*(t), s^*(t)) \in \Gamma(t)$$

i.e. $\dot{k}^*(t) = g(t, k^*(t), s^*(t))$ and $\zeta(t) \leqq w(t, k^*(t), s^*(t))$.

Therefore

$$\int_0^T \zeta(t)dt = \gamma,$$

and we can conclude that the pair $(k^*(t), s^*(t))$ is optimal.

Theorem. *Under Assumptions* 1–3, *the problem* (P) *has an optimal solution.*

References

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