81. A Note on Quasilinear Evolution Equations. II

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§1. **Introduction.** In this note we prove local existence and analyticity in t of solutions to quasilinear evolution equations

(1.1) $$du/dt+A(t,u)u=f(t,u), \qquad 0<t\leqq T,$$

(1.2) $$u(0)=u_0.$$

The unknown, u, is a function of t with values in a Banach space X. For fixed t and $v\in X$, the linear operator $-A(t,v)$ is the generator of an analytic semigroup in X and $f(t,v)\in X$.

We consider the equation (1.1) under the assumptions that the domain $D(A(t,v)^h)$ of $A(t,v)^h$ is independent of t, v for some $h>0$ and $A(t,A_0^{-\alpha}v)^h$ is the Hölder-continuous in v in the sense that

$$\|A(t,A_0^{-\alpha}v)^h A(t,A_0^{-\alpha}w)^{-h}-I\|\leqq C|v-w|^\eta,$$

while in the previous paper [1] we discussed it in the case that $A(t,A_0^{-\alpha}v)^h$ is the Lipschitz-continuous.

We use the same notations as in [1].

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§2. **Assumptions.** We first define $a\in X$. We shall make the following assumptions:

a-1) There exist $h=1/m$, where m is an integer, $m\geqq 2$, and $0\leqq\alpha<h/2$ such that $A_0^{-\alpha}$ is a well-defined bounded linear operator from X to X and $u_0\in D(A_0^{1+\alpha})$ where $A_0\equiv A(0,u_0)$.

a-2) There exists $T_0>0$, such that $A_{u_0}(t)\equiv A(t,u_0)$ is a well-defined operator from X to X for each $t\in[0,T_0)$.

a-3) For any $t\in[0,T_0)$ the resolvent of $A_{u_0}(t)$ contains the left half-plane and there exists C_1 such that $\|(\lambda-A_{u_0}(t))^{-1}\|\leqq C_1(1+|\lambda|)^{-1}$, $\operatorname{Re}\lambda\leqq 0$, and the domain, $D(A_{u_0}(t))$, of $A_{u_0}(t)$ is dense in X.

a-4) The domain $D(A_{u_0}(t)^h)=D$ of $A_{u_0}(t)^h$ is independent of $t\in[0,T_0)$ and there exist C_2, C_3, σ, $1-h+\alpha<\sigma\leqq 1$ such that

$$\|A_{u_0}(t)^h A_{u_0}(s)^{-h}\|\leqq C_2 \qquad t,s\in[0,T_0),$$

$$\|A_{u_0}(t)^h A_{u_0}(s)^{-h}-I\|\leqq C_3|t-s|^\sigma \qquad t,s\in[0,T_0).$$

a-5) $f_{u_0}(t)\equiv f(t,u_0)$ is defined and belongs to X for each $t\in[0,T_0)$, $f_{u_0}(0)\in D(A^h)$ and there exists C_4 such that

$$\|f_{u_0}(t)-f_{u_0}(s)\|\leqq C_4|t-s|^\sigma \qquad t,s\in[0,T_0).$$

These constants $C_i (i=1,2,3,4)$ do not depend on t, s. Then we can apply Kato's results [3]. It follows from Kato's theorem that there is a unique solution of

$$(\sharp)\qquad \begin{cases} d\hat{u}/dt + A_{u_0}(t)\hat{u} = f_{u_0}(t) \\ \hat{u}(0) = u_0. \end{cases}$$

Set

$$a = \frac{d^+}{dt} A_0^\alpha \hat{u}(t)|_{t=0}, \tag{2.1}$$

where $\hat{u}$ is the solution of ($\sharp$).

In the following $\Sigma(\phi\,;T) \equiv \{t \in \boldsymbol{C}\,; |\arg t| < \phi,\ 0 \leqq |t| < T\}$ is a sector in the complex plane.

Next we shall make the following assumptions with a;

A-1) = a-1).

A-2) A_0^{-1} is a completely continuous operator from X to X.

A-3) There exist $R>0$, $T_0>0$, $M>0$ and $\phi_0>0$ such that $A(t, A_0^{-\alpha}w)$ is a well-defined linear operator from X to X for each $t \in \Sigma(\phi_0\,; T_0)$ and $w \in N \equiv \{w \in X\,; \|w - A_0^{-\alpha}u_0\| < R\} \cap Y \cup \{A_0 u_0\}$, where

$$Y \equiv \bigcup_{t>0} \{v \in X\,; \|v - (A_0^\alpha u_0 + ta)\| < tM\} (0 < M \leqq \|a\|).$$

A-4) For any $t \in \Sigma(\phi_0\,; T_0)$ and $w \in N$

$$\begin{cases} \text{the resolvent of } A(t, A_0^{-\alpha}w) \text{ contains the left half-plane and} \\ \text{there exists } C_1 \text{ such that } \|(\lambda - A(t, A_0^{-\alpha}w))^{-1}\| \leqq C_1(1+|\lambda|)^{-1},\ \mathrm{Re}\,\lambda \\ \leqq 0, \text{ and the domain, } D(A(t, A_0^{-\alpha}w)), \text{ of } A(t, A_0^{-\alpha}w) \text{ is dense in } X. \end{cases} \tag{2.2}$$

A-5) The domain $D(A(t, A_0^{-\alpha}w)^h) = D$ of $A(t, A_0^{-\alpha}w)^h$ is independent of $t \in \Sigma(\phi_0\,; T_0)$ and $w \in N$.

A-6) There exist C_2, C_3, σ, $1-h+\alpha<\sigma\leqq 1$, $\alpha<\alpha''<h/2$, $(1-h+\alpha'')/(1-\alpha)<\eta<1$ such that

$$\|A(t, A_0^{-\alpha}w)^h A(s, A_0^{-\alpha}v)^{-h}\| \leqq C_2 \quad t, s \in \Sigma(\phi_0\,; T_0), \quad w, v \in N. \tag{2.3}$$

$$\|A(t, A_0^{-\alpha}w)^h A(s, A_0^{-\alpha}v)^{-h} - I\| \leqq C_3\{|t-s|^\sigma + \|w-v\|^\eta\} \quad t, s \in \Sigma(\phi_0\,; T_0), \quad w, v \in N. \tag{2.4}$$

A-7) $f(t, A_0^{-\alpha}w)$ is defined and belongs to X for each $t \in \Sigma(\phi_0\,; T_0)$ and $w \in N$, and there exists C_4 such that

$$\|f(t, A_0^{-\alpha}w) - f(s, A_0^{-\alpha}v)\| \leqq C_4\{|t-s|^\sigma + \|w-v\|^\eta\} \quad t, s \in \Sigma(\phi_0\,; T_0), \quad w, v \in N. \tag{2.5}$$

A-8) The map $\Phi: (t, w) \longmapsto A(t, A_0^{-\alpha}w)^h A_0^{-h}$ is analytic from $(\Sigma(\phi_0\,; T_0)\backslash\{0\}) \times (N\backslash\{A_0^\alpha u_0\})$ to $B(X)$.

A-9) The map $\Psi: (t, w) \longmapsto f(t, A_0^{-\alpha}w)$ is analytic from $(\Sigma(\phi_0\,; T_0)\backslash\{0\}) \times (N\backslash\{A_0^\alpha u_0\})$ into X.

These constants $C_i (i=1, 2, 3, 4)$ do not depend on t, s, v, w.

§ 3. **The main results.** We first restrict t to be real.

Theorem 1 (local existence). *Let the assumptions* A-1)–A-7) *hold with* $[0, T_0)$ *instead of* $\Sigma(\phi_0\,; T_0)$. *Then there exists* S_1, $0<S_1\leqq T_0$, *such that there exists at least one continuously differentiable solution of* (1.1) *for* $0<t<S_1$ *that is continuous for* $0\leqq t<S_1$ *and satisfies* (1.2).

Remark. In the case $h=1$, Sobolevskii [5] proved same results under similar assumptions to ours.

Theorem 2 (analyticity in t). *Let the assumptions* A-1)–A-9) *hold. Then there exist* T, $0<T\leqq T_0$, ϕ, $0<\phi<\phi_0$, $K>0$, k, $1-h<k<1$ *and at least one continuous function* u *mapping* $\Sigma(\phi\,;T)$ *into* X *such that* $u(0)=u_0$, $u(t)\in D(A(t,u(t)))$ *and* $\|A_0^\alpha u(t)-A_0^\alpha u_0\|<R$ *for* $t\in\Sigma(\phi\,;T)\backslash\{0\}$, $u:\Sigma(\phi\,;T)\backslash\{0\}\to X$ *is analytic*, $du(t)/dt+A(t,u(t))u(t)=f(t,u(t))$ *for* $t\in\Sigma(\phi\,;T)\backslash\{0\}$, *and* $\|A_0^\alpha u(t)-A_0^\alpha u_0\|\leqq K|t|^k$ *for* $t\in\Sigma(\phi\,;T)$.

The sketch of the proofs are given in § 4. The complete proofs of our results will be published elsewhere.

§ 4. Sketch of proofs. *Proof of Theorem* 1. Let $\zeta\in((1-h+\alpha'')/\eta,\ 1-\alpha)$, $0<\varepsilon<1$ and $L>0$. We consider the set $F(s)$ of all functions $v(t)$, defined on $[0,S)$, which satisfy the following:

$$v(0)=A_0^\alpha u_0,$$
$$\|v(t_1)-v(t_2)\|\leqq L|t_1-t_2|^\zeta \qquad \text{for any } t_1,\ t_2\in[0,S),$$
$$\|v(t)-(A_0^\alpha u_0+ta)\|\leqq Mt(1-\varepsilon) \qquad \text{for } t\in[0,S).$$

Then for sufficiently small positive S and for all $t\in[0,S)$, we get $v(t)\in N$ for any function $v(t)\in F(S)$. Hence the operator $A_v(t)=A(t,A_0^{-\alpha}v(t))$ is well defined for $t\in[0,S)$. Set $f_v(t)=f(t,A_0^{-\alpha}v(t))$ and $w_{v,\alpha}(t)=A_0^\alpha w_v(t)$, where w_v is the unique solution of

$$\begin{cases} dw_v/dt+A_v(t)w_v=f_v(t) & t\in[0,S),\\ w_v(0)=u_0. \end{cases}$$

Then using the linear theory of Kato [3] and some estimates in [2], we get $w_{v,\alpha}\in F(S)$ for sufficiently small S.

We define a transformation $T:v\mapsto w_{v,\alpha}$ for $v\in F(S)$. Then T maps $F(S)$ into itself. We now consider $F(S)$ as a subset of the Banach space $\tilde{Y}\equiv C([0,S);X)$ consisting of all the continuous functions $v(t)$ from $[0,S)$ into X with norm $|||v|||=\sup_{0\leqq t<S}\|v(t)\|$. Then T is a continuous operator in $F(S)$ with the topology induced by $\tilde{Y}$. From the assumption A-2), we obtain that the set $TF(S)$ is contained in a compact subset of $\tilde{Y}$. Therefore, by the Schauder's fixed point theorem there exists a fixed point v in $F(S):Tv=v$. Then $u=A_0^{-\alpha}v$ is a solution of (1.1), (1.2).

Proof of Theorem 2. From (2.2) there are constants C_5, $\phi_1>0$, $T_1>0$ such that for $t\in\Sigma(\phi_1\,;T_1)$, $w\in N$ and $|\theta|<\phi_1$ the resolvent of $e^{i\theta}A(t,A_0^{-\alpha}w)$ contains the left half-plane and

$$\|(\lambda-e^{i\theta}A(t,A_0^{-\alpha}w))^{-1}\|\leqq C_5(1+|\lambda|)^{-1} \qquad \mathrm{Re}\,\lambda\leqq 0.$$

We let $\phi=\min\{\phi_0,\phi_1\}$. We consider the set $E(S)$ of all functions $\tilde{v}(t)$, defined on $\Sigma(\phi\,;S)$, which satisfy the following:

$$\tilde{v}:\Sigma(\phi\,;S)\backslash\{0\}\longrightarrow X \text{ is analytic},$$
$$\tilde{v}(0)=A_0^\alpha u_0,$$
$$\|\tilde{v}(t)-\tilde{v}(0)\|\leqq L|t|^\zeta \qquad \text{for any } t\in\Sigma(\phi\,;S),$$

$$\|\tilde{v}(t_1)-\tilde{v}(t_2)\| \leqq L|t_1-t_2|^{\zeta} \qquad \text{for any real } t_1,\ t_2 \in [0, S),$$
$$\|\tilde{v}(t)-(A_0^{\alpha}u_0+ta)\| \leqq M\,|t|\,(1-\varepsilon) \qquad \text{for } t \in \Sigma(\phi\,;S).$$

Then, in the same way as in the proof of Theorem 1 using $\tilde{v} \in E(S)$, we can prove that $\tilde{w}_{\tilde{v},\alpha} \in E(S)$ for sufficiently small S, where $\tilde{w}_{\tilde{v},\alpha}(t)=A_0^{\alpha}\tilde{w}_{\tilde{v}}(t)$ and $\tilde{w}_{\tilde{v}}$ is the unique solution of

$$\begin{cases} d\tilde{w}_{\tilde{v}}/dt+A_{\tilde{v}}(t)\tilde{w}_{\tilde{v}}=f_{\tilde{v}}(t), & t \in \Sigma(\phi\,;S), \\ \tilde{w}_{\tilde{v}}(0)=u_0. \end{cases}$$

Next, we consider the set $F_0(S)$ of all functions $v(t)$ defined on $[0, S)$ such that for any $t \in [0, S)$ $v(t)=\tilde{v}(t)$ for some $\tilde{v} \in E(S)$. We define a transformation $\tilde{T}: \tilde{v} \mapsto \tilde{w}_{\tilde{v},\alpha}$ for $\tilde{v} \in E(S)$. Then $\tilde{T}$ maps $E(S)$ into itself. Using the operator $\tilde{T}$ we define a transformation $T: F_0(S) \to F_0(S)$ with $(Tv)(t)=(\tilde{T}\tilde{v})(t)$ for $t \in [0, S)$. We now consider $F_0(S)$ as a subset of $\tilde{Y} \equiv C([0, S)\,;X)$. Therefore there exist a fixed point $v \in F_0(S)$ such that $Tv=v$ and $\tilde{v} \in E(S)$ such that $\tilde{v}(t)=v(t)$ for $t \in [0, S)$. By the analyticity of $\tilde{v}$ we get $\tilde{T}\tilde{v}=\tilde{v}$. Putting $u=A_0^{-\alpha}\tilde{v}$, we can easily prove that u satisfies the conclusions of Theorem 2.

References

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