95. *Flat Coördinate System for the Deformation of Type* E_6

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Introduction. Let $f_{E_6}(x)=x^4+y^3+z^2$ be the defining equation of the simple singularity of type E_6. Let $F(x,t)$ be a versal deformation of $f_{E_6}(x)$ with the parameter space $S=\{(t_2,t_5,t_6,t_8,t_9,t_{12})\,;\,t_i\in \boldsymbol{C}\}$.

$$F(x,t)=f_{E_6}(x)+t_2x^2y+t_5xy+t_6x^2+t_8y+t_9x+t_{12}.$$

The purpose of this article is to determine the flat coördinate system in the space S. We refer the reader to [1]–[5] for basic notion and notation. We note also that in [6] the same result as (1.2) is obtained with different method.

1. Construction of flat coördinate system. First note that

$$(\mathrm{Hess}\,(F)-4t_2\partial F/\partial y)/6=12x^2y+2t_6y+(-4t_2t_8+t_5^2).$$

(1.1) We determine residue pairs for $e_i=[\partial F/\partial t_i\cdot dx\wedge dy\wedge dz]\in\varphi_*\Omega^3_{X/S}$ (see [3], [5]). Set $12\langle e_i,e_j\rangle=e(i,j)$. The result is as follows.

$$e(2,12)=e(5,9)=e(6,8)=1,\qquad e(2,9)=e(5,8)=e(5,6)=0,$$
$$e(6,6)=-t_2/2,\qquad e(2,8)=e(5,5)=t_2^2/6,$$
$$e(2,6)=-t_6/2-t_2^3/12,\qquad \mathrm{e}(2,5)=t_2t_5/4,$$
$$e(2,2)=t_2t_8/6-t_2^2t_6/6+t_5^2/12-t_2^5/72.$$

(1.2) By the method in [5], $m_{i,j}(t)=\langle dt_i,dt_j\rangle$ are determined.

$m_{2,j}=jt_j,\qquad j=2,5,6,8,9,12.$

$m_{5,5}=8t_8-4t_2t_6-t_2^4/6,\qquad m_{5,6}=9t_9-t_2^2t_5/2,$

$m_{5,8}=-t_2t_9/2-3t_5t_6/2-t_2^3t_5/12,\qquad m_{5,9}=12t_{12}+t_2^2t_8/3-3t_6^2-t_2^3t_6/6+t_2t_5^2/6,$

$m_{5,12}=-3t_6t_9/2-t_2^3t_9/12+t_2t_5t_8/6,$

$m_{6,6}=-10t_2t_8/3-2t_2^2t_6/3-5t_5^2/3,\qquad m_{6,8}=12t_{12}-4t_2^2t_8/3+7t_2t_2^2/12,$

$m_{6,9}=-4t_2^2t_9/3-13t_5t_8/3+7t_2t_5t_6/6,\qquad m_{6,12}=-2t_2^2t_{12}+7t_2t_5t_9/12-8t_8^2/3,$

$m_{8,8}=6t_2t_{12}-7t_5t_9/2+4t_6t_8-t_2^2t_5^2/24,$

$m_{8,9}=-3t_6t_9/2-7t_2t_5t_8/6-t_2^2t_5t_6/12+5t_5^3/12,$

$m_{8,12}=6t_6t_{12}-9t_9^2/4-t_2^2t_5t_9/24-4t_2t_8^2/3+5t_5^2t_8/12,$

$m_{9,9}=-2t_2^2t_{12}-5t_2t_5t_9/3-8t_8^2/3+8t_2t_6t_8/3-t_2^2t_6^2/6+4t_5^2t_6/3,$

$m_{9,12}=-3t_2t_5t_{12}+5t_2t_9t_8/6-t_2^2t_6t_9/12+5t_5^2t_9/12+t_5t_6t_8/2,$

$m_{12,12}=-2t_2t_8t_{12}-t_5^2t_{12}-t_2^2t_9^2/24+11t_5t_8t_9/6-4t_6t_8^2/3.$

(1.3) A flat coordinate system $\{s_i\}$ and a usual coordinate $\{t_i\}$ are transformed each other by the following rule.

$s_2=t_2,\quad s_5=t_5,\quad s_6=t_6+t_2^3/24,\quad s_8=t_8-t_2t_6/4-5t_2^4/576,\quad s_9=t_9+t_2^2t_5/12,$

$s_{12}=t_{12}+t_2^2t_8/24-t_6^2/8-5t_2^3t_6/288+t_2t_5^2/24-t_2^6/2^83^2,$

$t_2=s_2,\quad t_5=s_5,\quad t_6=s_6-s_2^3/24,\quad t_8=s_8+s_2s_6/4-s_2^4/576,\quad t_9=s_9-s_2^2s_5/12,$

$t_{12}=s_{12}-s_2^2s_8/24+s_6^2/8-s_2^3s_6/288-s_2s_5^2/24.$

(1.4) The relation between $\{s_i\}$ and a flat generator system $\{y_i\}$ determined in [2] is given by the following. Those constants are determined by comparing $\langle ds_i, ds_j\rangle$ and $\langle dy_i, dy_j\rangle$.

$$y_2=-s_2,\quad y_5=\sqrt{6}\,s_5/4,\quad y_6=-3s_6,\quad y_8=-3s_8,\quad y_9=3\sqrt{6}\,s_9/4,$$
$$y_{12}=-9s_{12}.$$

(1.5) Owing to the above identification we can describe the parameter space of the deformation in terms of J.S. Frame's invariants A, B, C, H, J, K (see [2]).

$$t_2=-A,\quad t_5=2\sqrt{6}\,B/3,\quad t_6=-C/3+A^3/12,\quad t_8=-H+AC/6-A^4/48,$$
$$t_9=2\sqrt{6}\,J/9-\sqrt{6}\,A^2B/18,$$
$$t_{12}=-K+A^2H/36+C^2/36-7A^3C/432+2AB^2/9+A^6/864,$$
$$A=-s_2,\quad B=\sqrt{6}\,s_5/4,\quad C=-3s_6-s_2^3/8,\quad H=-3s_8+3s_2s_6/4+s_2^4/192,$$
$$J=3\sqrt{6}\,s_9/4,\quad K=-9s_{12}-3s_2^2s_8+9s_6^2/8+s_2^3s_6/32-3s_2s_5^2/8.$$

2. Free deformations derived from $F(x, t)$. We remark some deformations given by restricting the parameter space.

(2.1) Let $D(s_2, s_5, s_6, s_8, s_9, s_{12})$ be the defining equation of the discriminant locus of the deformation of type E_6 normalized as $D(0', s_{12}) = s_{12}^6$. We note that $D(s)$ is an irreducible weighted homogeneous polynomial of weight 72.

(2.2) Set $s_5=s_9=0$ (or equivalently $t_5=t_9=0$). Then we get the deformation of type F_4 associated with the folding $W(E_6)\nearrow\!\searrow W(F_4)$. $D(s_2, 0, s_6, s_8, 0, s_{12})=g(s)g^*(s)^2$ where

$$g(s)=(s_{12}-s_2^2s_8/24+s_6^2/8-s_2^3s_6/2^5 3^2)^2+\frac{4}{27}(s_8+s_2s_6/4-s_2^4/2^6 3^2)^3,$$

$$g^*(s)=g(s_2, -s_6, s_8, -s_{12}).$$

The defining equation of the discriminant of type F_4 is given by $g(s)g^*(s)$. See [2], [4], [8], [9].

(2.3) Set $s_5=s_6=s_8=s_9=0$. Then the resulting deformation is of type $I_2(12)_E$ associated with the folding $W(E_6)\nearrow\!\searrow W(I_2(12))$.

$$F_{I_2(12)_E}(x, s)=x^4+y^3+s_2x^2y-s_2^3x^2/24-s_2^4y/576+s_{12},$$
$$D(s_2, 0, 0, 0, 0, s_{12})=(s_{12}^2-s_2^{12}/2^{16}3^9)^3.$$

(2.4) Three free deformations associated with unitary reflection groups are known.

No. 5 $\quad x^4+y^3+t_6x^2+t_{12},$
No. 8 $\quad x^4+y^3+t_8y+t_{12},$
No. 25 $\quad x^4+y^3+t_6x^2+t_9x+t_{12}.$

(2.5) There are two interesting deformations. One is related to the folding $W(F_4)\nearrow\!\searrow W(I_2(8))$. See Fig. below. The other is the deformation associated with the unitary reflection group No. 12, and the discriminant is a (3,4)-cusp.

$I_2(8)$ $\quad x^4+y^3+s_2x^2y-s_2^3x^2/24+(s_8-s_2^4/576)y-s_2^2s_8,$
No. 12 $\quad x^4+y^3+s_6x^2+s_8y+s_6^2/8.$

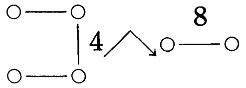

Fig. (2.6). Folding $W(F_4)\nearrow\searrow W(I_2(8))$.

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