109. *A Fundamental Conjecture on Exponent Semigroups*

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1. **Introduction.** Let S be a semigroup and let N be the multiplicative semigroup of all positive integers. The subset

$$E(S)=\{n\in N\mid (xy)^n=x^ny^n \text{ for all } x,\ y\in S\}$$

of N forms a subsemigroup of N and is called the *exponent semigroup* of S (Tamura [13]). If $m\in E(S)$ for some $m\geqq 2$, we say S is an *E-m semigroup*. The structure of E-m semigroups has been studied by Nordahl [11] and Cherubini and Varisco [2], and the structure of E-m groups was described by Alperin [1]. However, the structure of $E(S)$ itself had been veiled until a recent date. Only recently, Clarke, Pfiefer and Tamura [4] proved that if $2\in E(S)$, $E(S)$ is equal to either N or $N\backslash\{3\}$. Inspired by their work, Kobayashi [8] studied the case $3\in E(S)$ and has determined the structure of such $E(S)$ up to modulo 6.

Suggested by the results on the case $3\in E(S)$, we present in this note a fundamental conjecture which describes the structure of the exponent semigroups containing m modulo $m(m-1)$. We give several results supporting the validity of the conjecture. Above all, the conjecture is true for all finite semigroups. Most of the results will be given in [9] with complete proofs.

2. **Reduction to mod $m(m-1)$.** The following two theorems tell us that in studying the structure of $E(S)$ for an E-m semigroup S, it is essential to consider it modulo $m(m-1)$.

Theorem 1. *Let S be an E-m semigroup. If $k\in E(S)$ for some integer $k\geqq m$, then $\alpha m(m-1)+k\in E(S)$ for all integers $\alpha\geqq 0$.*

Corollary (Cherubini and Varisco [2]). *If S is an E-m semigroup, then $\alpha m(m-1)+m\in E(S)$ for all integers $\alpha\geqq 0$.*

Theorem 2. *Let S be an E-m semigroup. If $k\in E(S)$ for some integer $k\geqq 1$, then $\alpha m(m-1)+k\in E(S)$ for all integers $\alpha\geqq 2$.*

Corollary (cf. [8, Lemma 5]). *If S is an E-m semigroup, then $\alpha m(m-1)+1\in E(S)$ for all integers $\alpha\geqq 2$.*

It has been shown by Tamura [14] that for any integer $m\geqq 2$, there is an E-m semigroup S such that $E(S)$ does not contain $m(m-1)+1$. However, the following problem is still open: *For any E-m semigroup S, does $k(\geqq 2)\in E(S)$ imply $m(m-1)+k\in E(S)$?* When $m\leqq 3$, the answer is positive by [4] and [8], and when $k\geqq m$, Theorem 1 gives a positive answer. We can prove that the problem has an affirmative answer if $m\leqq 7$ or $k\leqq 4$. The first hurdle we cannot clear is the case $m=8$, $k=6$: *Does* 6, $8\in E(S)$ *imply* $62\in E(S)$?

3. Fundamental conjecture. Let S be an E-m semigroup. In view of Theorem 2, we define the subset $\bar{E}_m(S)$ of $Z_{m(m-1)}$, the residue class ring modulo $m(m-1)$ of the integers Z, by

$$\bar{E}_m(S)=\{\bar{n}\in Z_{m(m-1)}\mid n\in E(S)\},$$

where $\bar{n}$ denotes the class of n modulo $m(m-1)$. We call $\bar{E}_m(S)$ the *exponent semigroup* mod $m(m-1)$ of S. For an integer $n\geqq 1$ we define the subsets $M(n)$ and $N(n)$ of N by

$$M(n)=\{kn+1,\ kn+n\mid k=0,1,2,\cdots\}\quad\text{and}$$
$$N(n)=\{kn+1\mid k=0,1,2,\cdots\},$$

and the subsets $\bar{M}_m(n)$ and $\bar{N}_m(n)$ of $Z_{m(m-1)}$ by

$$\bar{M}_m(n)=\{\bar{k}\mid k\in M(n)\}\quad\text{and}\quad \bar{N}_m(n)=\{\bar{k}\mid k\in N(n)\}.$$

Conjecture. A subset $\bar{E}$ of $Z_{m(m-1)}$ $(m\geqq 2)$ is an exponent semigroup mod $m(m-1)$ of some E-m semigroup if and only if

$$(\sharp)\qquad \bar{E}=\bigcap_{i=1}^{s}\bar{M}_m(n_i)\cap\bar{N}_m(n)$$

for a finite number of integers $n_1,\cdots,n_s\geqq 2$ and $n\geqq 1$ such that $n_i\mid m$ or $n_i\mid(m-1)$ for $i=1,\cdots,s$, and $n\mid(m-1)$.

The results [8] on E-3 semigroups support the conjecture. Cherubini and Varisco [3] have shown that the conjecture is true for $m\leqq 9$. The "if" part of the conjecture is true (see Theorem 4 in § 4). So, we say "Conjecture is true for S" to mean "$\bar{E}=\bar{E}_m(S)$ is expressible in the form ($\sharp$) for $m(\geqq 2)\in E(S)$". This expression is not ambiguous because if $\bar{E}_m(S)$ is expressible in the form ($\sharp$) for some $m(\geqq 2)\in E(S)$, then it is so expressible for every $m(\geqq 2)\in E(S)$.

4. Main results. Following Petrich [12], we call a semigroup S *separative*, if $x^2=xy$ and $y^2=yx$ imply $x=y$, and $x^2=yx$ and $y^2=xy$ imply $x=y$, for any $x, y\in S$.

Theorem 3. *Let S be a separative semigroup. Then $E(S)$ is equal to either* $\{1\}$ *or*

$$(\sharp\sharp)\qquad \bigcap_{i=1}^{s}M(n_i)$$

for a finite number of integers $n_1,\cdots,n_s\geqq 2$. Conversely, for any subset E of N of the form ($\sharp\sharp$), there is a finite group G such that $E=E(G)$.

Corollary. *Let S be a separative semigroup. If $E(S)$ contains integers $m_1,\cdots,m_r\geqq 2$ such that $(m_1(m_1-1),\cdots,m_r(m_r-1))=2$, then S is commutative.*

Corollary contains the following result which is well-known for groups (see Herstein [7, p. 31]). *A separative semigroup S is commutative, if $E(S)$ contains three consecutive positive integers.* Recently, Corollary has been used to get a commutativity theorem for rings by

Kobayashi [10].

Remark. Since an E-m inverse semigroup is separative by [11, Corollary 1.12], the same conclusions as in Theorem 3 and its corollary hold for inverse semigroups.

Using Theorem 3 and [11, Proposition 1.6], we can prove

Theorem 4. *Let S be a 0-simple semigroup. Then $E(S)$ is equal to either $\{1\}$ or*

$$(\#\#\#) \qquad \bigcap_{i=1}^{s} M(n_i) \cap N(n)$$

for a finite number of integers $n_1, \cdots, n_s \geqq 2$ and $n \geqq 1$. Conversely, for any subset E of N of the form (###), there is a completely simple finite semigroup S such that $E = E(S)$.

By the second assertion in Theorem 4, we can say that every type of exponent semigroups considered in Conjecture comes from completely simple finite semigroups.

To prove that Conjecture is true for finite semigroups, we need to study the exponent semigroups of two special types of ideal extensions.

Proposition. *Let S be an ideal extension of a null semigroup N by a semigroup T with 0. If S is an E-m semigroup, then $\bar{E}_m(S) = \bar{E}_m(T)$.*

Theorem 5. *Let S be an ideal extension of a [0-]simple semigroup U by a semigroup T with 0. Then, either $E(S) = \{1\}$ or*

$$E(S) = E(U) \cap E(T) \cap N(l)$$

for some positive integer l.

The proof of Theorem 5 is carried out by a calculation using the normalized expression of the translational hull of a completely simple semigroup due to Clifford and Petrich [5]. By induction, utilizing Proposition and Theorem 5, on the length of principal series (see [6, p. 73] for the definition of principal series), we can prove

Corollary 1. *Conjecture is true for semigroups with principal series, especially for finite semigroups.*

A semigroup S is called *semisimple*, if every principal factor of S is [0-]simple (see [6, p. 74]). The following is a generalization of the first assertion in Theorem 4.

Corollary 2. *Let S be a semisimple semigroup satisfying the maximal condition on ideals. Then $E(S)$ is either equal to $\{1\}$ or expressible in the form (###) in Theorem 4.*

It seems probable that the conclusion in Corollary 2 is true for arbitrary semisimple semigroups. Utilizing this last corollary, we can prove

Theorem 6. *If S is a semilattice of simple semigroups, then $E(S)$ is either equal to $\{1\}$ or expressible in the form (###).*

Since an E-m regular semigroup is a semilattice of simple semigroups by [11, Corollary 1.11], we get

Corollary. *If S is a regular semigroup, then $E(S)$ is either equal to $\{1\}$ or expressible in the form* (###).

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