1. *On the Infinitesimal Generators and the Asymptotic Behavior of Nonlinear Contraction Semi-Groups*

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1. Introduction. Throughout this paper, let X be a Banach space, $A: D(A)(\subset X)\to X$ be a dissipative operator satisfying range condition

(R) $R(I-tA)\supset \overline{D(A)}$ (the closure of $D(A)$) for every $t>0$,

where I denotes the identity, $J_t=(I-tA)^{-1}$ for $t>0$, $\hat{D}(A)=\{x\in\overline{D(A)}: \lim_{t\to 0+}\|J_tx-x\|/t<\infty\}$, and let $\{T(t): t\geqq 0\}$ be the nonlinear contraction semi-group on $\overline{D(A)}$ generated by A, i.e., $T(t)x=\lim_{\lambda\to 0+} J_\lambda^{[t/\lambda]}x$ for $x\in\overline{D(A)}$ and $t\geqq 0$, where [ ] denotes the Gaussian bracket (see [2]). We define $|Ax|$, $d(0, R(A))$, $|||Ax|||$ and A^0 by $|Ax|=\lim_{t\to 0+}\|J_tx-x\|/t$ for $x\in\hat{D}(A)$, $d(0, R(A))=\inf\{\|x\|: x\in R(A)$ (the range of A)$\}$, $|||Ax|||=\inf\{\|y\|: y\in Ax\}$ for $x\in D(A)$ and $A^0x=\{y\in Ax: \|y\|=|||Ax|||\}$, respectively.

The purpose of this paper is to prove the following theorems.

Theorem 1. *Suppose that X^* (the dual of X) has Fréchet differentiable norm. Then we have the following:* (i) *For each $x\in\hat{D}(A)$, $\lim_{t\to 0+} t^{-1}(T(t)x-x)$ and $\lim_{t\to 0+} t^{-1}(J_tx-x)$ both exist and are equal. Define A^* by $A^*x=\lim_{t\to 0+} t^{-1}(T(t)x-x)$ for $x\in\hat{D}(A)$. Then A^* is the infinitesimal generator of $\{T(t): t\geqq 0\}$.* (ii) *$(\overline{A})^0$ is single-valued, $D((\overline{A})^0)=D(\overline{A})=\hat{D}(A)$ and $(\overline{A})^0=A^*$, where $\overline{A}$ denotes the closure of A.*

Theorem 2. *Suppose that X^* has Fréchet differentiable norm. Then we have the following:* (i) *There exists an $x_0\in X$ such that $\lim_{t\to\infty} t^{-1}T(t)x=\lim_{t\to\infty} t^{-1}J_tx=x_0$ for all $x\in\overline{D(A)}$.* (ii) *x_0 is the unique point of least norm in $\overline{R(A)}$.*

Theorem 1 generalizes Plant's results [6, Theorems 2 and 5]. Plant proved (i) in Theorem 1 under the assumption that X is uniformly convex, and (ii) under the assumption that X is uniformly convex and X^* is strictly convex. Theorem 2 generalizes Reich's result [7, Theorem 3.3]. Reich proved (i) and (ii) in Theorem 2 under the assumption that X is uniformly convex, or X^* has Fréchet differentiable norm and X is (UG).

2. Lemmas. The following was proved in [1]:

Lemma 1. *$\hat{D}(A)=\{x\in\overline{D(A)}: \lim_{t\to 0+}\|T(t)x-x\|/t<\infty\}$, and $\lim_{t\to 0+}\|T(t)x-x\|/t=|Ax|$ $(\equiv\lim_{t\to 0+}\|J_tx-x\|/t)$ for every $x\in\hat{D}(A)$.*

The following lemma is due to Plant [6, (2.10)].

Lemma 2. *Let $x \in \overline{D(A)}$. Then for every $s, t>0$*

$$\|T(s)x-J_tx\| \leqq (1-s/t)\|J_tx-x\|+(2/t)\int_0^s \|T(r)x-x\|\,dr.$$

Lemma 3. *Let $x \in \overline{D(A)}$. Then* $\lim_{t\to\infty}\|T(t)x\|/t=\lim_{t\to\infty}\|J_tx\|/t$ $=d(0, R(A))$.

Proof. It is known that $\lim_{t\to\infty}\|J_tx\|/t=d(0,R(A))$ (see [7, Lemma 2.1]). Let $v \in R(A)$. Then there is a $u \in D(A)$ such that $v \in Au$. Since $u=J_\lambda(u-\lambda v)$ for $\lambda>0$ and each J_λ is a contraction (i.e., $\|J_\lambda y-J_\lambda z\| \leqq \|y-z\|$ for $y, z \in D(J_\lambda)$), we have

(1) $\quad \|J_\lambda^i x-u\| \leqq \|J_\lambda^{i-1}x-u\|+\lambda\|v\| \quad$ for $\lambda>0$ and $i \geqq 1$.

Let $t>\lambda>0$ and add (1) for $i=1,2,\cdots,[t/\lambda]$. Then $\|J_\lambda^{[t/\lambda]}x-u\| \leqq \|x-u\| +t\|v\|$. Letting $\lambda\to 0+$, we have that $\|T(t)x-u\| \leqq \|x-u\|+t\|v\|$ for $t>0$ and then $\limsup_{t\to\infty}\|T(t)x\|/t \leqq \|v\|$. Hence $\limsup_{t\to\infty}\|T(t)x\|/t \leqq d(0,R(A))$. By Lemma 2 and $\|J_tx-x\|-\|T(s)x-x\| \leqq \|T(s)x-J_tx\|$,

$$\|T(s)x-x\| \geqq (s/t)\|J_tx-x\|-(2/t)\int_0^s \|T(r)x-x\|\,dr$$

for $t, s>0$. Letting $t\to\infty$, $\|T(s)x-x\| \geqq d(0,R(A))s$ for $s>0$ and hence $\liminf_{s\to\infty}\|T(s)x\|/s \geqq d(0,R(A))$. This completes the proof.

3. Proof of Theorems. It is known that X^* has Fréchet differentiable norm if and only if X is reflexive, and strictly convex and has the following property (A). (See [3].)

(A) If $w\text{-}\lim_{n\to\infty} x_n=x$ and $\lim_{n\to\infty}\|x_n\|=\|x\|$, then $\lim_{n\to\infty} x_n=x$.

Here $w\text{-}\lim_{n\to\infty} x_n$ denotes the weak limit of $\{x_n\}$.

Let $x \in \overline{D(A)}$, and let $f(\cdot):(0,\infty)\to X^*$ be a function such that $f(t) \in F(t^{-1}(J_tx-x))$ for $t>0$, where $F(u)=\{u^* \in X^*:(u,u^*)=\|u\|^2=\|u^*\|^2\}$ for $u \in X$ and (u,u^*) denotes the value of u^* at u. By the resolvent identity, $\|J_tx-J_sx\|=\|J_s((s/t)x+(1-s/t)J_tx)-J_sx\| \leqq (1-s/t)\|J_tx-x\|$ for $t>s>0$. Combining this with $\mathrm{Re}(J_sx-x, f(t)) \geqq \|J_tx-x\|^2/t -\|J_sx-J_tx\|\,\|J_tx-x\|/t$, we have that $\mathrm{Re}(s^{-1}(J_sx-x), f(t)) \geqq \|J_tx-x\|^2/t^2$ for $t>s>0$, where $\mathrm{Re}(u,u^*)$ denotes the real part of (u,u^*). By $\mathrm{Re}(T(\sigma)x-x, f(t)) \geqq \|J_tx-x\|^2/t-\|T(\sigma)x-J_tx\|\|J_tx-x\|/t$ and Lemma 2,

$$\begin{aligned}&\mathrm{Re}(\sigma^{-1}(T(\sigma)x-x), f(t))\\ &\quad \geqq \|J_tx-x\|^2/t^2-(2/t^2)\|J_tx-x\|(1/\sigma)\int_0^\sigma \|T(r)x-x\|\,dr\end{aligned}$$

for $t, \sigma>0$. Consequently we have

$$\begin{aligned}&\mathrm{Re}(s^{-1}(J_sx-x)+\sigma^{-1}(T(\sigma)x-x), f(t))\\ &\quad \geqq 2\|J_tx-x\|^2/t^2-(2/t^2)\|J_tx-x\|(1/\sigma)\int_0^\sigma \|T(r)x-x\|\,dr \qquad (2)\\ &\qquad\qquad \text{for } t>s>0 \text{ and } \sigma>0.\end{aligned}$$

Proof of Theorem 1. (i) Let $x \in \hat{D}(A)$, and let $\{s_k\}$ and $\{\sigma_k\}$ be sequences of positive numbers such that $s_k\to 0$ and $\sigma_k\to 0$ as $k\to\infty$. Since X is reflexive and $\lim_{s\to 0+}\|T(s)x-x\|/s=\lim_{s\to 0+}\|J_sx-x\|/s=|Ax|$ $<\infty$ by Lemma 1, there exist $u, v \in X$ and $\{k_i\}$, $\{k'_i\}$ (subsequences of

$\{k\}$) such that $w\text{-}\lim_{i\to\infty} s_{k_i}^{-1}(J_{s_{k_i}}x-x)=u$ and $w\text{-}\lim_{i\to\infty} \sigma_{k'_i}^{-1}(T(\sigma_{k'_i})x-x)=v$. Putting $s=s_{k_i}$, $\sigma=\sigma_{k'_i}$ in (2) and letting $i\to\infty$, we have

(3) $\qquad \mathrm{Re}\,(u+v,\ f(t))\geqq 2\,\|J_t x-x\|^2/t^2 \qquad$ for $t>0$.

Since X^* is reflexive and $\|f(t)\|=\|J_t x-x\|/t$, there exists an $f\in X^*$ and a sequence $\{t_n\}$, $t_n>0$, with $\lim_{n\to\infty} t_n=0$ such that $w\text{-}\lim_{n\to\infty} f(t_n)=f$. Therefore by (3)

(4) $\qquad \mathrm{Re}\,(u+v,\ f)\geqq 2\,|Ax|^2.$

Noting that $\|u\|\leqq|Ax|$, $\|v\|\leqq|Ax|$ and $\|f\|\leqq|Ax|$, it follows from (4) that $\|u+v\|=\|u\|+\|v\|$ and $\|u\|=\|v\|=|Ax|$. So, by strict convexity of X, we have that $u=v$. Consequently, $w\text{-}\lim_{s\to 0+} s^{-1}(J_s x-x)$ and $w\text{-}\lim_{\sigma\to 0+} \sigma^{-1}(T(\sigma)x-x)$ both exist and $w\text{-}\lim_{s\to 0+} s^{-1}(J_s x-x)=w\text{-}\lim_{\sigma\to 0+} \sigma^{-1}(T(\sigma)x-x)=v$. Moreover,

$$\lim_{s\to 0+} \|J_s x-x\|/s=\lim_{\sigma\to 0+} \|T(\sigma)x-x\|/\sigma=|Ax|=\|v\|.$$

Since X has the property (A), we obtain $\lim_{s\to 0+} s^{-1}(J_s x-x)=v=\lim_{\sigma\to 0+} \sigma^{-1}(T(\sigma)x-x)$. It follows from Lemma 1 that A^* is the infinitesimal generator of $\{T(t): t\geqq 0\}$. (The infinitesimal generator A_0 of the semi-group is defined by $A_0 z=\lim_{h\to 0+} h^{-1}(T(h)z-z)$ whenever the limit exists.) (ii) Note that $\overline{A}$ is a closed dissipative operator and $(I-t\overline{A})^{-1}x=J_t x$ for $x\in\overline{D(A)}$ and $t>0$. Since $\|J_t x-x\|/t=\|(I-t\overline{A})^{-1}x-x\|/t\leqq|||\overline{A}x|||$ for $x\in D(\overline{A})$ and $t>0$, we have that $D((\overline{A})^0)\subset D(\overline{A})\subset\hat{D}(A)$. Let $x\in\hat{D}(A)$. Then $t^{-1}(J_t x-x)\in AJ_t x\subset\overline{A}J_t x$ for $t>0$,

$$\lim_{t\to 0+} J_t x=x \quad\text{and}\quad \lim_{t\to 0+} t^{-1}(J_t x-x)=A^*x.$$

The closedness of $\overline{A}$ implies that $x\in D(\overline{A})$ and $A^*x\in\overline{A}x$. But $\|A^*x\|\leqq|||\overline{A}x|||$ by $\|J_t x-x\|/t\leqq|||\overline{A}x|||$. Consequently, $x\in D((\overline{A})^0)$ and $A^*x\in(\overline{A})^0x$. Therefore $D((\overline{A})^0)=D(\overline{A})=\hat{D}(A)$ and $A^*\subset(\overline{A})^0$. To show that $(\overline{A})^0=A^*$, let $x\in D((\overline{A})^0)$ and $z\in(\overline{A})^0x$. Since $t^{-1}(J_t x-x)\in\overline{A}J_t x$, the dissipativity of $\overline{A}$ implies

$$\|J_t x-x-\lambda(t^{-1}(J_t x-x)-z)\|\geqq\|J_t x-x\| \qquad \text{for } \lambda>0 \text{ and } t>0.$$

Put $\lambda=t/2$. Then we have $\|t^{-1}(J_t x-x)+z\|\geqq 2\,\|J_t x-x\|/t$ for $t>0$. Letting $t\to 0+$, $\|A^*x+z\|\geqq 2\,\|A^*x\|$ and hence $\|A^*x+z\|=2\,\|A^*x\|=\|A^*x\|+\|z\|$. By strict convexity of X, $z=A^*x$. This completes the proof.

Remark 1. The proof of Theorem 1 (i) shows that if X is reflexive and strictly convex, then for every $x\in\hat{D}(A)$ $w\text{-}\lim_{t\to 0+} t^{-1}(T(t)x-x)$ and $w\text{-}\lim_{t\to 0+} t^{-1}(J_t x-x)$ both exist and are equal.

Proof of Theorem 2. Put $d=d(0,R(A))$ $(=d(0,\overline{R(A)}))$ and let $x\in\overline{D(A)}$. (i) Since $\|f(t)\|=\|J_t x-x\|/t\to d$ as $t\to\infty$ (by Lemma 3), there exists an $f\in X^*$ and a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n=\infty$ such that $w\text{-}\lim_{n\to\infty} f(t_n)=f$. By (2) we get

(5) $\qquad \mathrm{Re}\,(s^{-1}(J_s x-x)+\sigma^{-1}(T(\sigma)x-x),\ f)\geqq 2d^2 \qquad$ for $s,\sigma>0$.

Let $\{s_k\}$ and $\{\sigma_k\}$ be sequences such that $s_k\to\infty$ and $\sigma_k\to\infty$ as $k\to\infty$. Since $\lim_{s\to\infty} \|T(s)x-x\|/s=\lim_{s\to\infty} \|J_s x-x\|/s=d$ by Lemma 3, there exist $u, v \in X$ and $\{k_i\}$, $\{k'_i\}$ (subsequences of $\{k\}$) such that w-$\lim_{i\to\infty} s_{k_i}^{-1}(J_{s_{k_i}}x-x)=u$ and w-$\lim_{i\to\infty} \sigma_{k'_i}^{-1}(T(\sigma_{k'_i})x-x)=v$. Then by (5) we have that $\mathrm{Re}\,(u+v, f)\geqq 2d^2$. Using the same argument in the proof of Theorem 1, we see that $\lim_{t\to\infty} t^{-1}T(t)x$ and $\lim_{t\to\infty} t^{-1}J_tx$ both exist and are equal. Put $x_0=\lim_{t\to\infty} t^{-1}J_tx$. Since $T(t)$ and J_t are contractions, $\lim_{t\to\infty} t^{-1}T(t)z=\lim_{t\to\infty} t^{-1}J_tz=x_0$ for all $z\in \overline{D(A)}$. (ii) It is easy to see that x_0 is a point of least norm in $\overline{R(A)}$. We now prove the uniqueness. Let $y\in D(A)$ and $z\in Ay$. Since A is dissipative, $\|J_tx-y-\lambda(t^{-1}(J_tx-x)-z)\|\geqq\|J_tx-y\|$ for $\lambda, t>0$. Put $\lambda=t/2$. Then we have $\|t^{-1}J_tx+z+t^{-1}(x-2y)\|\geqq 2\|J_tx-y\|/t$ for $t>0$. Letting $t\to\infty$, $\|x_0+z\|\geqq 2d$. Consequently, $\|x_0+w\|\geqq 2d$ for every $w\in\overline{R(A)}$. In particular, let $w\in\overline{R(A)}$ and $\|w\|=d$. Then $\|x_0+w\|=\|x_0\|+\|w\|=2d$. By strict convexity of X, $w=x_0$. This completes the proof.

Remark 2. It follows from the proof of Theorem 2 (i) that if X is reflexive and strictly convex then there exists an $x_0\in X$ such that w-$\lim_{t\to\infty} t^{-1}T(t)x=w$-$\lim_{t\to\infty} t^{-1}J_tx=x_0$ for every $x\in\overline{D(A)}$.

Corollary ([4]). *Let C be a closed convex subset of $X, T: C\to C$ be a contraction and $x\in C$.* (i) *If X^* has Fréchet differentiable norm, then $\{n^{-1}T^nx\}$ is convergent to the unique point of least norm in $\overline{R(T-I)}$.* (ii) *If X is reflexive and strictly convex, then $\{n^{-1}T^nx\}$ is weakly convergent.*

Proof. Put $A=T-I$. Then A is a dissipative operator satisfying (R). Let $\{T(t): t\geqq 0\}$ be the contraction semi-group generated by A. It is known that $\|T(n)x-T^nx\|\leqq\sqrt{n}\,\|Tx-x\|$ for $n\geqq 1$ (see [5]). Now, the results follow from Theorem 2 and Remark 2.

Added in Proof. 1. Recently Prof. Reich informed me that he has obtained (i) in Theorems 1 and 2, and (ii) under an additional assumption that X is smooth. (See S. Reich "A note on the asymptotic behavior of nonlinear semigroups and the range of accretive operators, MRC Technical Summary Report # 2198 (1981)".)

2. Let $\tilde{A}$ be a maximal dissipative operator in $\overline{D(A)}$ such that $\tilde{A}\supset A$. If X is reflexive and strictly convex then $(\tilde{A})^0$ is the weak infinitesimal generator of $\{T(t): t\geqq 0\}$.

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