52. Calculus on Gaussian White Noise. IV

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In the previous parts of this series [11], [12], [16], we have given a foundation of calculus on Gaussian white noise and shown the relation between our formulation and Hida's one [1], [2]. In this part, we will treat two kinds of infinite dimensional Laplacians Δ_V and Δ_L related to Gaussian white noise. In the following, we will use the same notations and definitions as in § 5 of Part II.

§ 12. Laplacians Δ_V and Δ_L. We have defined operators ∂_t, $t \in T$, in Part II. By Theorem 6.1, we can see that an operator

$$\Delta_V \equiv \int_T d\nu(t)\partial_t\partial_t \tag{12.1}$$

is well defined and continuous on $\mathcal{H}$ and that its dual is given by

$$\Delta_V^* \equiv \int_T d\nu(t)\partial_t^*\partial_t^*. \tag{12.2}$$

We call the operator Δ_V *Volterra's Laplacian*. By (6.1) in Part II, the operator $\tilde{\Delta}_V \equiv \mathcal{S}\Delta_V\mathcal{S}^{-1}$ on $\mathcal{F}$ is expressed in the form

$$\tilde{\Delta}_V U(\xi) = \int_T d\nu(t)\frac{\delta}{\delta\xi(t)}\frac{\delta}{\delta\xi(t)}U(\xi). \tag{12.3}$$

Lévy [17] introduced another Laplacian for functionals of a special type as follows,

$$\Delta\left\{\int_T f_1(u)\xi(u)^2 d\nu(u) + \int_{T\times T} f_2(u,v)\xi(u)\xi(v)d\nu(u)d\nu(v)\right\} \equiv 2\int_T f_1(u)d\nu(u). \tag{12.4}$$

Motivated by this, Hida has introduced a Lapalacian for generalized Brownian functionals. We will give an extension of his definition.

Lemma 12.1. *Let $V(\xi)$ be in $\mathcal{F}^*$, then there exists an element $V^{(n)}(\xi; t_1, \cdots, t_n) \in \mathcal{E}^{*\hat{\otimes}n}$, such that*

$$V^{(n)}(\xi; \eta_1, \cdots, \eta_n) = \langle V^{(n)}(\xi; \cdot), \eta_1\hat{\otimes}\cdots\hat{\otimes}\eta_n\rangle.$$

Take $n=2$ as a special case. Then $V^{(2)}(\xi; t, s)$ is in $\mathcal{E}^{*\hat{\otimes}2}$. Suppose that the generalized function $V^{(2)}(\xi; \cdot)$ is a signed measure on T^2 for fixed ξ. Then we can restrict the measure onto the diagonal set $D \equiv \{(t,s) \in T^2; t=s\}$. We define *Lévy's Laplacian* $\tilde{\Delta}_L V$ as the mass of D with respect to the signed measure denoted by

$$\tilde{\Delta}_L V(\xi) \equiv \int_D V^{(2)}(\xi; t, s)d\nu^2(t,s). \tag{12.5}$$

For a Ψ in $\mathcal{H}^*$, $\Delta_L\Psi$ is defined by

(12.6) $$\Delta_L\Psi \equiv \mathcal{S}^{-1}\tilde{\Delta}_L\mathcal{S}\Psi,$$

if $\tilde{\Delta}_L(\mathcal{S}\Psi)$ is well defined and it belongs to $\mathcal{F}^*$. Denote the domain of Δ_L by $\mathcal{D}(\Delta_L)$. The following theorem is obtained by Theorem 6.2 and Remark 4.6.

Theorem 12.2. (i) *Δ_V is a continuous operator on $\mathcal{H}$ satisfying*

$$\|\Delta_V^k\varphi\|_{\mathcal{H}^{(p)}} \leq \sqrt{(2k)!}(\|\delta\|\rho^q)^{2k}(1-\rho^{2q})^{-k+1/2}\|\varphi\|_{\mathcal{H}^{(p+q)}},$$

(ii) *Δ_V^* is a one-to-one continuous operator on $\mathcal{H}^*$ satisfying*

$$\mathcal{S}(\Delta_V^*\Psi)(\xi) = \|\xi\|_0^2\mathcal{S}\Psi(\xi) \qquad \text{for } \Psi \in \mathcal{H}^*,$$

(iii) *if $\nu(T) < \infty$ and if ψ is in $(L^2) = \mathcal{H}^{(0)}$, then $\Delta_V^*\psi$ is in $\mathcal{D}(\Delta_L)$ and satisfies*

$$\frac{1}{2\nu(T)}\Delta_L\Delta_V^*\psi = \psi \quad \text{and} \quad \Delta_L\psi = 0.$$

By the theorem, we can define a one-parameter group of operators

(12.7) $$\exp[\tau\Delta_V] \equiv \sum_{k=0}^{\infty}\frac{\tau^k}{k!}\Delta_V^k.$$

Actually we have

(12.8) $$\|\exp[\tau\Delta_V]\varphi\|_{\mathcal{H}^{(p)}} \leq 2(1-\rho^{2q})\|\varphi\|_{\mathcal{H}^{(p+q)}}$$

for sufficiently large q as $2(1+2|\tau|\|\delta\|^2)\rho^{2q} < 1$.

Theorem 12.3. (i) *$\{\exp[\tau\Delta_V]; \tau \in R\}$ is a one-parameter group of continuous operators on $\mathcal{H}$,*

(ii) *for $\varphi \in \mathcal{H}$, $\exp[\tau\Delta_V]\varphi$ is analytic in $\tau \in R$.*

Proposition 12.4. *We have the following formulae;*

(i) $\Delta_V \exp[\langle x, \eta\rangle] = \|\eta\|_0^2 \exp[\langle x, \eta\rangle]$,
$\exp[\tau\Delta_V]\exp[\langle x, \eta\rangle] = \exp[\langle x, \eta\rangle + \tau\|\eta\|_0^2]$,

(ii) $\Delta_V H_n(\langle x, \eta\rangle; \|\eta\|_0^2) = n(n-1)\|\eta\|_0^2 H_{n-2}(\langle x, \eta\rangle; \|\eta\|_0^2)$,
$\exp[\tau\Delta_V]H_n(\langle x, \eta\rangle; \|\eta\|_0^2) = H_n(\langle x, \eta\rangle; (1-2\tau)\|\eta\|_0^2)$,

(iii) $\Delta_V\langle x, \eta\rangle^n = n(n-1)\langle x, \eta\rangle^{n-2}$,
$\exp[\tau\Delta_V]\langle x, \eta\rangle^n = H_n(\langle x, \eta\rangle; -2\tau\|\eta\|_0^2)$,

(iv) $\exp[-\Delta_V/2]\langle x, \eta\rangle^n = H_n(\langle x, \eta\rangle; \|\eta\|_0^2)$.

Theorem 12.5. (i) *Let φ be in $\mathcal{H}$. Then $(\mathcal{S}\varphi)(\xi)$ can be extended to a continuous functional $(\mathcal{S}\varphi)(x)$ on $\mathcal{H}^*$, which satisfies*

$$\exp[\Delta_V/2]\varphi(x) = (\mathcal{S}\varphi)(x),$$

and hence $\varphi(x)$ has a continuous version $\mathcal{S}(\exp[-\Delta_V/2]\varphi)(x)$.

(ii) *Let U be in $\mathcal{F}$. Then the continuous extension $U(x)$ of U on $\mathcal{E}^*$ belongs to $\mathcal{H}$ and satisfies*

$$(\exp[-\Delta_V/2]U(x))\cdot = :U(x)\cdot: \quad \text{and} \quad \mathcal{S}(\exp[-\Delta_V/2]U)(\xi) = U(\xi).$$

Remark 12.6. By the theorem, we can think of that $\mathcal{H}$ is a family of continuous functionals on $\mathcal{E}^*$. In this sense, $\mathcal{F}$ coincides with the set of all restrictions of elements of $\mathcal{H}$ to $\mathcal{E}$. We have that for $\varphi(x) \in \mathcal{H}$,

(12.9) $$(\partial_t\varphi)|_{\mathcal{E}} = \delta/\delta\xi(t)(\varphi|_{\mathcal{E}}) \quad \text{and} \quad (\Delta_V\varphi)|_{\mathcal{E}} = \tilde{\Delta}_V(\varphi|_{\mathcal{E}}),$$

(12.10) $$|\varphi(x+\lambda\delta_t)-\varphi(x)-\lambda\partial_t\varphi(x)|=o(\lambda).$$

Corollary 12.7. *Put* $\varphi(x)=A^*(f_n)1$ *for* $f_n\in\mathcal{E}^{\hat{\otimes}n}$. *Then we have* $\langle x^{\hat{\otimes}n},f_n\rangle=\sum_{k=0}^{[n/2]}\frac{1}{k!}\left(\frac{\Delta_V}{2}\right)^k\varphi$ *and* $\varphi(x)=\sum_{k=0}^{[n/2]}\frac{1}{k!}\left(-\frac{\Delta_V}{2}\right)^k\langle x^{\hat{\otimes}n},f_n\rangle$.

§13. Expressions of Δ_V by coordinates. Let $\{\zeta_k\}$ be a c.o.n.s. of $E_0=L^2(T,\nu)$. Then for $\zeta\in\mathcal{E}$,

(13.1) $$\xi=\sum_k^\infty\langle\zeta_k,\xi\rangle\zeta_k$$

converges in E_0 and hence also in $\mathcal{E}^*$. Since $U(\xi)\in\mathcal{F}$ can be extended to a continuous functional on $\mathcal{E}^*$ as seen in §7, we can define a function $U(\xi^1,\cdots,\xi^k,\cdots)\equiv U(\xi)$ for $(\xi^1,\cdots,\xi^k,\cdots)$ with $\xi=\sum\xi^k\zeta_k\in\mathcal{E}^*$. By Theorem 3.3, $U^{(2)}(\xi\,;\zeta_1,\zeta_2)$ can be extended to a continuous linear functional on $\mathcal{E}^{*\hat{\otimes}2}$. Then we get

(13.2) $$U^{(2)}(\xi\,;\zeta_i,\zeta_j)=\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\xi^j}U(\xi^1,\cdots,\xi^i,\cdots,\xi^j,\cdots).$$

Theorem 13.1. *For* $U(\xi)\in\mathcal{F}$ *and for any c.o.n.s. of* E_0, *it holds that for* $\xi=\sum\xi^k\zeta_k$

$$\tilde{\Delta}_VU(\xi)=\sum_{k=1}^\infty U^{(2)}(\xi\,;\zeta_k,\zeta_k).$$

We now suppose the following assumption:

(S) There exists a c.o.n.s. $\{\zeta_k\}$ of E_0 which is also a c.o.g.s. of E_p for every p.

Then a sequence of projections Π_N, $N\geq1$, is defined by

(13.3) $$\Pi_Nx=\sum_{k=1}^N\langle x,\zeta_k\rangle\zeta_k$$

for $x\in\mathcal{E}^*$, since $\{\zeta_k\}$ is included in $\mathcal{E}$.

Remark 13.2. For f_n in $E_p^{\hat{\otimes}n}$, $\Pi_N^{\hat{\otimes}n}f_n$ is in $E_p^{\hat{\otimes}n}$ and converges to f_n in $E_p^{\hat{\otimes}n}$ as $N\to\infty$.

Theorem 13.3. *Assume Assumption* (S). *Then for* $\varphi(x)\in\mathcal{H}$ *and* $\tau>0$, *the following hold;*

(i) $\varphi(\Pi_Nx)\to\varphi(x)$ *in* $\mathcal{H}$ *and pointwisely for* $x\in\mathcal{E}^*$,

(ii) $\Delta_V\varphi(x)=\lim_{N\to\infty}\sum_{k=1}^N\frac{\partial}{\partial X^k}\frac{\partial}{\partial X^k}\varphi\left(\sum_{k=1}^N X^k\zeta_k\right)$, *for* $x=\sum_{k=1}^\infty X^k\zeta_k$,

(iii) $\exp[\tau\Delta_V/2]\varphi(x)$

$$=\lim_{N\to\infty}(2\pi\tau)^{-N/2}\int_{R^N}\exp\left[\frac{-1}{2\tau}\sum_{k=1}^N(X^k-a^k)^2\right]\varphi\left(\sum_{k=1}^N a^k\zeta_k\right)da^1\cdots da^N.$$

The corresponding assertions are true for the space $\mathcal{F}$.

Now let us introduce a class of entire functions for $m>0$ by

(13.11) $$\mathcal{A}_m^{Re}\equiv\left\{h(z)=\sum_{n=0}^\infty a_nz^n\,;\,a_n\text{'s are reals and }\lim_{n\to\infty}(n\,!)^m\,|a_n|^2=0\right\}.$$

Theorem 13.4. *Suppose that* $f_n\in\mathcal{E}^{\hat{\otimes}n}$, $h(z)\in\mathcal{A}_m^{Re}$ *and* $\varphi(x)$ *is either* $\langle x^{\hat{\otimes}n},f_n\rangle$ *or* $A^*(f_n)1$. *Then we have*

(i) $h(\varphi(x))$ *belongs to* $\mathcal{H}$,

(ii) $\Delta_V h(\varphi(x)) = h'(\varphi(x))\Delta_V\varphi(x) + h''(\varphi(x))\int_T d\nu(t)(\partial_t\varphi)^2$,

(iii) $\exp[-\Delta_V/2]h(\langle x^{\hat{\otimes} n}, f_n\rangle) \xrightarrow{S} h(\langle \xi^{\hat{\otimes} n}, f_n\rangle)$.

Example 13.5. For $h \in \mathcal{A}_1^{Re}$ and $\tau > 0$, we obtain

$$\exp[\tau\Delta_V/2]h(\langle x, \eta\rangle) = (2\pi\tau\|\eta\|_0^2)^{-1/2}\int h(z)\exp\left[\frac{-1}{2\tau\|\eta\|_0^2}(\langle x, \eta\rangle - z)^2\right]dz.$$

Example 13.6. The function $\exp[z]$ does not belong to $\mathcal{A}_1^{Re}$, but we can have the following. For $f_2 \in \mathcal{E}^{\hat{\otimes} 2}$, we can find a c.o.n.s. $\{\eta_k\}$ such that $f_2 = \sum \rho_k \eta_k \hat{\otimes} \eta_k$, $\sum |\rho_k| < \infty$. If $|\tau|$ is so small as

$$4(1+|\tau|\,\|\delta\|)\,\|f_2\|_{E_p^{\hat{\otimes} 2}} < 1, \tag{13.12}$$

then for $\zeta \in \mathcal{E}$,

$$\exp[\tau\Delta_V/2]\exp[-\langle x^{\hat{\otimes} 2}, f_2\rangle/2 + \langle x, \zeta\rangle] = \prod_k (1+\tau\rho_k)^{-1}\exp\left[\frac{1}{1+\tau\rho_k}\left\{-\frac{\rho_k}{2}\langle x, \eta_k\rangle^2 + \langle x, \eta_k\rangle\langle \eta_k, \zeta\rangle + \frac{\tau}{2}\langle \eta_k, \zeta\rangle^2\right\}\right] \tag{13.13}$$

holds in $\mathcal{H}^{(p)}$.

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