# 119. Branching of Singularities for Degenerate Hyperbolic Operator and Stokes Phenomena. III

By Kazuo AMANO*) and Gen NAKAMURA**)

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1. This note is a continuation of our previous notes [1] and [2]. The aims of this note are to complete our previous result of [2] and to sharpen the results obtained by paraphrasing Shinkai's results [4] for a system to our single equation. The details and further discussions will appear in [3].

2. Assumptions and results. Let $t\in[-T,T]$, $x=(x_1,\cdots,x_n)\in \boldsymbol{R}^n$, $D_t=\partial/(\sqrt{-1}\partial t)$, $D_x=(D_1,\cdots,D_n)$, $D_j=\partial/(\sqrt{-1}\partial x_j)$ $(1\le j\le n)$ and $P\equiv P(t,X,D_t,D_x)$ be an $m$th order linear partial differential operator of the form:

$$P=\sum_{j=0}^{m}\sum_{i=0}^{m-j}P_{i,j}(t,X,D_x)D_t^{m-j-i},$$

where each $P_{i,j}(t,x,\xi)$ is a homogeneous polynomial of degree $i$ with respect to $\xi=(\xi_1,\cdots,\xi_n)\in\boldsymbol{R}^n$. For simplicity, we assume all the coefficients have bounded derivatives of any order on $[-T,T]\times\boldsymbol{R}^n_x$.

We assume the following conditions (A.1)–(A.3) for $P$ which are invariant under change of $x$ variable.

(A.1) $P_m(t,x,\tau,\xi)$ is smoothly factorizable as follows:

$$P_m(t,x,\tau,\xi)=\prod_{j=1}^{m}(\tau-t^{\ell}\lambda_j(t,x,\xi)),$$

where $\ell\in N$ and $\lambda_j(t,x,\xi)\in C^\infty([-T,T]\times\boldsymbol{R}^n\times(\boldsymbol{R}^n-\{0\}))$ $(1\le j\le n)$ are real valued.

(A.2) There exists a constant $C>0$ such that

$$|\lambda_j(t,x,\xi)-\lambda_k(t,x,\xi)|\ge C|\xi|$$

for any $j\ne k$ and $(t,x,\xi)$.

(A.3) Each $P_{i,j}(t,x,\xi)$ $(i\ell\ge j,\ m-j-i\ge 0)$ has the property:

$$P_{i,j}(t,x,\xi)=t^{i\ell-j}\tilde{P}_{i,j}(t,x,\xi)$$

where $P_{i,j}(t,x,\xi)$ is a homogeneous polynomial of degree $i$ in $\xi$ and its coefficients have bounded derivatives of any order on $[-T,T]\times\boldsymbol{R}^n_x$.

In order to state our results we need some notations and definitions.

Definitions (Phase functions and double phase functions). For each $j$ $(1\le j\le n)$, define a phase function $\phi_j(t,s,x,\xi)$ as the solution of the Cauchy problem:

$$\partial\phi_j/\partial t-t^{\ell}\lambda_j(t,x,\nabla_x\phi_j)=0,\qquad \phi_j|_{t=s}=x\cdot\xi$$

*) Department of Mathematics, Josai University.
**) Department of Mathematics, M.I.T. and Department of Mathematics, Josai University.

where $x\cdot\xi=\sum_{j=1}^{n} x_j\xi_j$ for $x=(x_1,\cdots,x_n)$, $\xi=(\xi_1,\cdots,\xi_n)$. Also, for each $j,k$ $(1\leq j,\ k\leq m)$, define a double phase function $\phi_{j,k}(t,s,x,\xi)$ as the solution of the Cauchy problem:

$$\partial_t\phi_{j,k}/\partial t-t^{\ell}\lambda_j(t,x,\nabla_x\phi_{j,k})=0,\qquad \phi_{j,k}|_{t=0}=\phi_k(0,s,x,\xi).$$

**Remark.** If we denote by $T_j(t,s)$ and $T_{j,k}(t,s)$ the homogeneous symplectic transformations corresponding to $\phi_j(t,s,x,\xi)$ and $\phi_{j,k}(t,s,x,\xi)$, then we have $T_{j,k}(t,s)=T_j(t,0)\circ T_k(0,s)$.

**Notations** (Indices $m_i^{\pm}$ related to the growth order of the amplitude). Put $\mu_i(x,\xi)=-H_i(x,\xi)/G_i(x,\xi)$, where

$$G_i(x,\xi)=\sum_{j=0}^{m-1}(m-j)\lambda_i(0,x,\xi)^{m-j-1}\tilde{P}_{j,0}(0,x,\xi),$$

$$H_i(x,\xi)=(\ell/2)\sum_{j=0}^{m-2}(m-j)(m-j-1)\lambda_i(0,x,\xi)^{m-j-1}\tilde{P}_{j,0}(0,x,\xi)$$
$$+\sqrt{-1}\sum_{j=1}^{m-1}\lambda_i(0,x,\xi)^{m-j-1}\tilde{P}_{j,1}(0,x,\xi).$$

Then define $m_i^{\pm}$ by $m_i^{\pm}=\sup_{(x,\xi)}\mathrm{Re}\,\{(\pm\mu_i(x,\xi))\}$.

**Definitions** (Central connection coefficients). Set

$$L_0=\sum_{j=0}^{m}\sum_{i\ell\geq j,\,m-j-i\geq 0}t^{i\ell-j}\tilde{P}_{i,j}(0,x,\xi)D_t^{m-j-i}.$$

Let $\exp(\sqrt{-1}(\ell+1)^{-1}t^{\ell+1}\lambda_i(0,x,\xi))V_i^{\pm}(t,x,\xi)$ $(1\leq i\leq m)$ be a fundamental system of solutions of $L_0$ in $\pm t>0$ with the property:

$$V_i^{\pm}(t,x,\xi)\simeq e_i^{*}(t,x,\xi)=t^{\mu_i(x,\xi)}\sum_{r=0}^{\infty}e_{i,r}(x,\xi)t^{-r}\qquad \text{as } t\to\pm\infty$$

where $e_{i,0}(x,\xi)\equiv 1$ and the symbol "$\simeq$" denotes the asymptotic expansion uniform with respect to the parameters $x\in\boldsymbol{R}^n$, $(|\xi|=1)$ which is also valid for the derivatives of $V_i^{\pm}(t,x,\xi)$. The asymptotic series for the derivatives of $V_i^{\pm}$ are obtained by differentiating $e_i^{*}$ formally.

We also define $V_{j,i-1}^{-}(t,x,\xi)$ and $\tilde{V}_{j,i-1}(t,x,\xi)$ by

$$V_{j,i-1}^{-}(t,x,\xi)=\exp(-\sqrt{-1}(\ell+1)^{-1}t^{\ell+1}\lambda_j(0,x,\xi))V_i^{-}(t,x,\xi)$$

$\tilde{V}_{j,i-1}(t,x,\xi)=$ the $(i,j)$-cofactor of matrix

$$\left(V_j^{-}(t,x,\xi)\,;\,\begin{matrix} i\downarrow 0,\cdots,m-1\\ j\rightarrow 1,\cdots,m\end{matrix}\right).$$

Furthermore we define $U_i(t,x,\xi)$ as a solution of the Cauchy problem: $L_0U_i=0$, $D_t^hU_i|_{t=0}=\delta_{h,i-1}$ $(0\leq h\leq m-1)$, where $\delta_{h,i}$ denotes Kronecker's delta. Then the central connection coefficients $T_{\pm}^{(i,j)}(x,\xi)$ $(1\leq i,\ j\leq m)$ are defined by the relations:

$$U_i(t,x,\xi)=\sum_{j=1}^{m}\exp(\sqrt{-1}(\ell+1)^{-1}t^{\ell+1}\lambda_j(0,x,\xi))T_{\pm}^{(i,j)}(x,\xi)V_j^{\pm}(t,x,\xi)$$

in $\pm t>0$ for $1\leq i\leq m$. In addition we define $\tilde{T}_{-}^{(i,j)}(x,\xi)$ as the $(i,j)$-cofactor of matrix

$$\left(T_{-}^{(i,j)}(x,\xi)\,;\,\begin{matrix} i\downarrow 1,\cdots,m\\ j\rightarrow 1,\cdots,m\end{matrix}\right).$$

**Definitions** (Symbol classes). Let $\mu,\kappa,\lambda\in\boldsymbol{R}$.

(1) $a(t,s,x,\xi)\in S^{\pm}[\mu]$ if $a(t,s,x,\xi)$ is $C^{\infty}$ in $\{0\leq\pm t\leq T_0\}\times\{0\leq\pm s\leq T_0\}\times\boldsymbol{R}_x^n\times(\boldsymbol{R}^n-\{0\})$ with the following property: For any $p,q\in\boldsymbol{Z}_+$, $\alpha,\beta\in\boldsymbol{Z}_+^n$, there exists $C>0$ such that

$$|D_t^pD_s^qD_x^{\alpha}D_{\xi}^{\beta}a(t,s,x,\xi)|\leq C(1+|\xi|)^{\mu-|\beta|}\qquad(|\xi|\geq 1).$$

(2) $a(t,x,\xi)\in\tilde{S}^{\pm}[\mu,\kappa]$ if $a(t,x,\xi)$ is $C^{\infty}$ in $\{0\leq\pm t\leq T_0\}\times\boldsymbol{R}_x^n\times(\boldsymbol{R}^n-\{0\})$ with the following property: For any $p\in\boldsymbol{Z}_+$, $\alpha,\beta\in\boldsymbol{Z}_+^n$, there

exists $C>0$ such that

$$|D_t^p D_x^\alpha D_\xi^\beta a(t,x,\xi)| \le C(1+|\xi|)^{\mu-|\beta|}(|\xi|^{-1}+|t|^{\ell+1})^{(\kappa-p)/(\ell+1)} \qquad (|\xi|\ge 1).$$

(3) $a(t,s,x,\xi)\in \tilde{S}^{\pm}[\mu,\kappa,\lambda]$ if $a(t,s,x,\xi)$ is $C^\infty$ in $\{0\le \pm t\le T_0\}\times\{0\le \pm s\le T_0\}\times R_x^n\times(R^n-\{0\})$ with the following property: For any $p,q\in Z_+$, $\alpha,\beta\in Z_+^n$, there exists $C>0$ such that

$$|D_t^p D_s^q D_x^\alpha D_\xi^\beta a(t,s,x,\xi)| \le C(1+|\xi|)^{\mu-|\beta|}(|\xi|^{-1}+|t|^{\ell+1})^{(\kappa-p)/(\ell+1)} \cdot(|\xi|^{-1}+|s|^{\ell+1})^{(\lambda-q)/(\ell+1)} \qquad (|\xi|\ge 1).$$

(4) $a(t,s,x,\xi)\in \tilde{\tilde{S}}^{-}[\mu,\kappa]$ if $a(t,s,x,\xi)$ is $C^\infty$ in $\{(t,s);-T_0\le s\le t\le 0\}\times R_x^n\times(R^n-\{0\})$ with the property: For any $p,q\in Z_+$, there exists $C>0$ such that

$$|D_t^p D_s^q D_x^\alpha D_\xi^\beta a(t,s,x,\xi)| \le C|t|^\kappa(1+|\xi|)^{\mu-|\beta|} \qquad (|\xi|\ge 1).$$

Here $Z_+$ denotes the set of non-negative integers and $Z_+^n=\{\alpha=(\alpha_1,\cdots,\alpha_n);\alpha_i\in Z_+\ (1\le i\le n)\}$. Moreover, we define the symbol classes $S^-[-\infty]$, $\tilde{S}^+[\mu,\infty]$, $\tilde{S}^-[\mu,\kappa,\infty]$, $\tilde{\tilde{S}}^-[\mu,\infty]$ by $S^-[-\infty]=\bigcap_\mu S^-[\mu]$, $\tilde{S}^+[\mu,\infty]=\bigcap_{\kappa>0} S^+[\mu,\kappa]$, $\tilde{S}^-[\mu,\kappa,\infty]=\bigcap_{\lambda>0} S^-[\mu,\kappa,\lambda]$, $\tilde{\tilde{S}}^-[\mu,\infty]=\bigcap_{\kappa>0}\tilde{\tilde{S}}^-[\mu,\kappa]$.

As a final step of our preparation to describe our theorem, let us clarify the definition of parametrix.

Definition. Corresponding to each $i$ $(1\le i\le m)$, we call $E_i^\pm(t,s)$ a parametrix if $E_i^\pm(t,s)g\in C^\infty(\Delta_\pm;\mathcal{D}'(R^n))$ for each $g\in\mathcal{E}'(R^n)$ and satisfies

$$PE_i^\pm\equiv 0 \quad\text{in}\quad s<t,\quad D_t^h E_i^\pm|_{t=s}\equiv\delta_{h,i-1}I \quad (0\le h\le m-1)$$

where $\Delta_\pm=\{(t,s)\in[-T_0,T_0]\times[-T_0,T_0];\ s\le t,\ \pm s,\ \pm t>0\}$ and the symbol "$\equiv$" stands for an equality modulo integral operator with $C^\infty$ kernel. In the case, $s,t$ vary over $\Delta=\{(t,s);-T_0\le s\le t\le T_0\}$, we also define parametrices $E_i(t,s)$ $(1\le i\le m)$ in the same way as we did for $E_i^\pm(t,s)$ $(1\le i\le m)$. The only modification is to replace $\Delta_\pm$ by $\Delta$.

**Theorem 1.** *There exist $T_0>0$ and symbols*

$a_{i,j}^+(t,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{S}^+[(\ell+1)^{-1}(m_j^+-i+1+\varepsilon),\ m_j^++\varepsilon]$,

$\tilde{a}_{i,j}^+(t,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{S}^+[(\ell+1)^{-1}(m_j^+-i+1+\varepsilon),\ \infty]$ $\quad(1\le i,\ j\le m)$,

$a_{i,j}^-(t,s,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{S}^-[2\varepsilon(\ell+1)^{-1}-(i-1),\ m_j^++\varepsilon,\ m_j^--\ell(i-1)+\varepsilon]$,

$\tilde{a}_{i,j}^-(t,s,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{S}^-[2\varepsilon(\ell+1)^{-1}-(i-1),\ m_j^++\varepsilon,\ \infty]$,

$\tilde{\tilde{a}}_{i,j}^-(t,s,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{\tilde{S}}^-[2\varepsilon(\ell+1)^{-1}-(i-1),\ \infty]$ $\quad(1\le i,\ j\le m)$

*such that parametrices $E_i^+(t,0)$ $(1\le i\le m)$ and $E_i^-(t,s)$ $(1\le i\le m)$ are given by*

$$(E_i^+(t,0)\cdot)(x)=(2\pi)^{-n}\sum_{j=1}^m Os-\iint \exp[\sqrt{-1}(\phi_j(t,0,x,\xi)-y\cdot\xi)]\chi(\xi)(a_{i,j}^+(t,x,\xi)+\tilde{a}_{i,j}^+(t,x,\xi))\cdot dyd\xi \qquad (0\le t\le T_0,\ x\in R^n)^{*)},$$

$$(E_i^-(t,s)\cdot)(x)=(2\pi)^{-n}\sum_{j=1}^m Os-\iint \exp[\sqrt{-1}(\phi_j(t,s,x,\xi)-y\cdot\xi)]\chi(\xi)(a_{i,j}^-(t,s,x,\xi)+\tilde{a}_{i,j}^-(t,s,x,\xi)+\tilde{\tilde{a}}_{i,j}^-(t,s,x,\xi))\cdot dyd\xi \qquad (-T_0\le s\le t\le 0,\ x\in R^n)$$

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*) The sign "$Os$" before the integral sign denotes the usual oscillatory integral.

*where* $\chi(\xi)\in C_0^\infty(\boldsymbol{R}^n)$; $\chi(\xi)=0$ $(|\xi|\leq 1/2)$, $\chi(\xi)=1$ $(|\xi|\geq 1)$. *In addition* $\tilde{a}_{i,j}^+$ $(1\leq i, j\leq m)$ *are flat at* $t=0$ *and* $\tilde{a}_{i,j}^-$ $(1\leq i, j\leq m)$ *are flat at* $s=0$. *Moreover,*

$$a_{i,j}^+(t,x,\xi)-T_+^{(i,j)}(x,\xi)V_j^+(t,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{S}^+[(\ell+1)^{-1}(m_j^+-i+1+\varepsilon), m_j^++1+\varepsilon],$$

$$a_{i,j}^-(t,s,x,\xi)-a_{i,j,0}^{-,0}(t,s,x,\xi)\in\bigcap_{\varepsilon>0}\tilde{S}^-[2\varepsilon(\ell+1)^{-1}-(i-1), m_j^++1+\varepsilon, m_j^--\ell(i-1)+\varepsilon]+\bigcap_{\varepsilon>0}\tilde{S}^-[2\varepsilon(\ell+1)^{-1}-(i-1), m_j^++\varepsilon, m_j^--\ell(i-1)+1+\varepsilon],$$

$$a_{i,j,0}^{-,0}(t,s,x,\xi)=\det(T_-^{(i,j)}(x,\xi))_{1\leq i,j\leq m}V_j^-(t,x,\xi)V_j^-(s,x,\xi).$$

**Theorem 2.** *Let* $s<0$ *be appropriately small. Then the following assertions hold.*

(1) $$(E(t,s)\cdot)(x)=(2\pi)^{-n}\sum_{\nu,\mu=1}^n Os-\iint\exp[\sqrt{-1}(\phi_{\nu,\mu}(t,s,x,\eta)-y\cdot\eta)]a_{i,\nu,\mu}(t,s,x,\eta)\cdot dyd\eta$$

*for* $t>0$. *Here, if* $s, t$ *are small enough, there exists* $R>0$ *such that the main part of*

$$a_{i,\nu,\mu}(t,s,x,\eta)=\sum_{j=1}^m T_+^{(j,\nu)}(x,\nabla_x\phi_{\nu,\mu}(t,s,x,\eta))\tilde{T}_-^{(j,\mu)}(\nabla_\eta\phi_{\nu,\mu}(t,s,x,\eta),\eta)\cdot V^+(t,x,\nabla_x\phi_{\nu,\mu}(t,s,x,\eta))\tilde{V}_{\mu,i-1}(s,\nabla_\eta\phi_{\nu,\mu}(t,s,x,\eta),\eta)\cdot(\text{nonzero factor})\quad \text{for } |\eta|\geq R.$$

(2) *Let* $i$ $(0\leq i\leq m-1)$ *be an integer and* $u_h\in\mathcal{E}'(\boldsymbol{R}^n)$ $(0\leq h\leq m-1)$ *whose wavefront sets* $WF(u_h)$ $(0\leq h\leq m-1)$ *satisfy* $\bigcup_{h\neq i}WF(u_h)=\phi$, $WF(u_i)=\{(y^0,\rho\eta^0);\rho>0\}$. *Let* $u(t,s,x)$ *be the solution of the Cauchy problem*: $Pu=0$, $D_t^h u|_{t=0}=u_h$ $(0\leq h\leq m-1)$. *Suppose the following condition* (♯) *holds for a particular pair* $(\nu_0,\mu_0)$ *and a sufficiently small* $s', t'$ $(s\leq s'<0<t')$

(♯) $$\sum_{j=1}^m T_+^{(j,\nu_0)}(T_{\nu_0,\mu_0}(t',s')\circ T_{\mu_0}(s',s)(y^0,\eta^0))\tilde{T}_-^{(j,\mu_0)}(T_{\mu_0}(s',s)(y^0,\eta^0))\neq 0.$$

*Then, for any* $t$ $(0\leq t\leq T_0)$, *the wavefront set* $WF(u(t,s))$ *of* $u(t,s)$ *contains* $T_{\nu_0,\mu_0}(t,s)(y^0,\eta^0)$.

**Remark.** (1) Since $T_{\mu_0}(0,0)(y,\eta)=T_{\nu_0,\mu_0}(0,0)(y,\eta)$, the following condition (♯)′ implies (♯).

(♯)′ $$\sum_{j=1}^m T^{(j,\nu_0)}(y^0,\eta^0)\tilde{T}_-^{(j,\mu_0)}(y^0,\eta^0)\neq 0.$$

The left hand side of (♯)′ is the so-called Stokes' multiplier.

(2) Our proofs of Theorems 1 and 2 provide many other conditions instead of (♯). (Consult [3].)

## References

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